CMPT409/815: Advanced Algorithms

Homework Assignment 5

Instructor: Igor Shinkar *Due date: December 9, 2020*

Instructions: Submit your solution to Coursys (if scanned, make sure it is good quality).

Question 1 (25 points) *Consider the following test for sortedness.*

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Require: Given an array A[1, 2, ..., n] of length n
Choose i, j ∈ [n] and k ∈ [n − 1] independently uniformly at random
Read A[i], A[j], A[k], A[k + 1]
if the 4 queries violate sortedness then
    return NOT SORTED
else
    return SORTED
```

Show an example of an array that fools this test with high probability. Specifically, for any n show an array A of length n that is at least $1/4$-far from any sorted array, but $\Pr[ALG(A) = SORTED] > 1 - O(1/\sqrt{n})$.

Question 2 (25 points) *Modify the algorithm for sortedness we saw in class so that it also works when A has equal values in it.*

Fourier analysis of the boolean functions

Question 3 (25 points) *Prove that if a boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is δ-close to some linear function L (for some $\delta \in (0, 1/2)$, then it is at least $(1/2 - \delta)$-far from all other linear functions.*

(Hint: Prove that for any two distinct linear functions L_1, L_2 it holds that $\Pr_{x\in\{0,1\}^n}[L_1(x) = L_2(x)] = 1/2$.)

Question 4 (25 points) *Let $f\colon \{0,1\}^n \to \{0,1\}$ be a boolean function, and let $C_{1/2+\delta}(f)$ be the set of all linear functions L such that $\Pr[f(x) = L(x)] > 1/2 + \delta$. Prove that $|C_{1/2+\delta}(f)| \le O(1/\delta^2)$ for all f and all $\delta \in (0, 0.1)$.*

(Hint: Look at the Fourier coefficients of f.)