CMPT409/815: Advanced Algorithms

Final Exam

Instructor: Igor Shinkar *December 13, 2019*

Name: ______________________

SFU ID: ______________________

- Write your name and SFU ID **clearly**.
- This is a closed book exam, no calculators, cell phones, or any other material.
- The exam consists of five (5) problems. The sum of all points in the five problems is 120.
- If you score more than 100, your grade will be 100.
- Write your answers in the provided space.
- There is an extra page at the end of the exam. You may use it if needed.
- Explain all your answers. Provide enough details to convince me that you understand the solution.

Question 1 (20 points) *Design a polynomial time algorithm that given a 4-colorable graph G on n vertices outputs a legal $\tilde{O}(n^{4/7})$ coloring of G.*

You may use as a subroutine an algorithm we saw in class that given a 3-colorable graph G finds a legal coloring of G using $\tilde{O}(\Delta^{1/3})$-colors, where Δ is the maximal degree of a vertex in G

Question 2 (25 points)

(a) **[15 points]** *In the* Minimum Vertex Cover *problem for* 3*-uniform hypergraphs the input is a* 3*-uniform hypergraph* $H = (U, F)$*, and the goal is to find a vertex cover* $C \subseteq U$ *of* H *such that* $|C|$ *is minimized.*[1] *Let* $\textsf{min-VC}(H) = \min\{|C| : C \textit{ is a vertex cover in } H\}$.

Design a polynomial time 3*-approximation algorithm for the problem of finding a minimum vertex cover in a hypergraph. That is, the algorithm must output a vertex cover* C *such that* $|C| \leq 3 \cdot \textsf{min-VC}(H)$.

(b) **[10 points]** *Given an undirected graph* $G = (V, E)$*, denote by* $Rem_{\triangle}(G)$ *the minimum number of edges required to remove from* G *so that the remaining graph contains no triangle.*

Design a polynomial time 3-approximation algorithm for $Rem_{\triangle}(G)$*. That is, the algorithm outputs a subset of edges* $E' \subseteq E$ *such that* $G = (V, E \setminus E')$ *contains no triangles, and* $|E'| \leq 3 \cdot Rem_{\triangle}(G)$.

[1] Recall, a *vertex cover* in $H = (U, F)$ is a collection of vertices $C \subseteq U$ that touches every hyperedge in H, i.e., for every $e \in F$ it holds that $e \cap C \neq \emptyset$.

Question 3 (30 points) *In the maximum directed cut problem (MAX-DICUT) the input is a directed graph $G = (V, E)$, and for each directed edge $(i \to j) \in E$ we have a non-negative weight $w_{(i\to j)} \in \mathbb{R}^+$.*

The goal is to partition V into two sets (U, W) such that and $W = V \setminus U$ so as to maximize the total weight of the arcs going from U to W (that is, $\sum_{\substack{(i\to j)\in E \\ i\in U, j\in W}} w_{i,j}$).

(a) **[10 points]** *Give a polynomial time randomized 1/4-approximation algorithm for the MAX-DICUT problem. That is the randomized algorithm returns a partition $V = (U, W = V \setminus U)$ such that the expected total weight of the arcs going from U to W is at least $1/4$ of the optimum.*

(b) **[5 points]** *Show that the following LP is a relaxation of the MAX-DICUT problem:*

$$\textit{maximize} \sum_{(i\to j)\in E} w_{i,j} z_{i,j}$$

$$\text{subject to} \begin{cases} z_{(i\to j)} \le x_i & \forall (i \to j) \in E \\ z_{(i\to j)} \le 1 - x_j & \forall (i \to j) \in E \\ 0 \le z_{(i\to j)} \le 1 & \forall (i \to j) \in E \\ 0 \le x_i \le 1 & \forall v \in V \end{cases}$$

(c) **[15 points]** *Consider the randomized rounding algorithm for MAX-DICUT that solves the LP above and puts each vertex i into U independently with probability $\frac{1}{4} + \frac{x_i}{2}$. Show that this gives a randomized 1/2-approximation algorithm for MAX-DICUT. That is, in the output the expected total weight of the arcs going from U to W is at least $1/2$ of the optimum.*

(Hint: Given a solution to LP relate the probability that edge $(i \to j)$ is in the cut (U, W) to the LP value of $z_{i,j}$.)

Question 4 (25 points) *In the* MAX-3-CNF *problem, we are given a collection* Φ *of clauses* $C_1, \ldots, C_m$ *over the variables* $x_1, \ldots, x_n$*, where each clause is a disjunction (OR) of three literals (i.e.* x_i *or* $\bar{x}_i$*). The value of* Φ *is defined as the maximal fraction of clauses that can be satisfies by an assignment. That is,*

$$val(\Phi) = \max_{x=(x_1,\ldots,x_n)\in\{0,1\}^n} \frac{|\{i \in [m] : C_i \text{ is satisfied by } x\}|}{m}.$$

Let Φ *be such* MAX-3-CNF *instance with* m *clauses over* n *variables such that* $m = n^2$*. Consider a random sub-formula* Φ' *of* Φ *obtained by keeping each clause of* Φ *with some probability* $p \in [0, 1]$*. (The sub-formula* Φ' *is over the same variables as* Φ*.)*

Fix $\varepsilon \in (0, 0.1)$*. Prove that there exists an absolute constant* $C > 0$ *such that if* Φ' *is obtained from* Φ *by keeping each clause with probability* $p \geq \frac{Cn}{\varepsilon^2 m}$*, then the following holds.*

(a) **[5 points]** *The number of clauses* m' *in* Φ' *satisfies* $\Pr[|m' - pm| < \varepsilon pm] > 1 - \exp(-\Omega(n))$.

(b) **[5 points]** *If* $val(\Phi) = 1$*, then* $\Pr[val(\Phi') = 1] = 1$.

(c) **[5 points]** $\Pr[val(\Phi') \geq val(\Phi) - \varepsilon] > 1 - \exp(-\Omega(n))$.

(d) **[10 points]** $\Pr[val(\Phi') \leq val(\Phi) + \varepsilon] > 1 - \exp(-\Omega(n))$.

You may want to use the following concentration inequality:
Chernoff bound: *Let* $X_1, X_2, ..., X_m$ *be independent Bernoulli random variables with* $\Pr[X_i = 1] = p_i$*. Let* $X = \sum X_i$*, and let* $\mu = \mathbb{E}[X] = \sum_i p_i$*. Then, for all* $\delta \in (0, 1)$ *we have* $\Pr[|X - \mu| \geq \delta\mu] \leq 2e^{-\frac{\delta^2\mu}{3}}$

Question 5 (20 points) *Given a Boolean function* $f\colon \{1,-1\}^n \to \{1,-1\}$, *define* $\sigma(f) = \sum_{i=1}^{n} \hat{f}(\{i\})$.

- **[6 points]** *Let* $w(x)\colon \{1,-1\}^n \to \mathbb{R}$ *be the real valued function defined as* $w(x) = \sum_{i=1}^{n} x_i$. *Prove that* $\sigma(f) = \mathbb{E}_x[f(x)w(x)]$.
- **[7 points]** *Prove that for odd* n *among all Boolean functions* $f\colon \{1,-1\}^n \to \{1,-1\}$ *the value* $\sigma(f)$ *is maximized by the* Majority *function, where* $Maj_n(x) = \text{sign}(\sum_{i=1}^{n} x_i)$.
- **[7 points]** *The influence of the i'th coordinate on* f *is defined as* $Inf_i(f) = \Pr_{x\{1,-1\}^n}[f(x) \neq f(x^{(i)})]$, *where* $x^{(i)}$ *is the vector obtained from* x *by flipping the ith coordinates. Prove that if* f *is a monotone function (i.e., if* $x_i \geq y_i$ *for all* $i \in [n]$, *then* $f(x) \geq f(y)$*), then* $Inf_i(f) = \hat{f}(\{i\})$.

  Conclude that among all Boolean monotone functions Maj_n *has the maximal total influence.*