CMPT404/705: Design and Analysis of Computing Algorithms

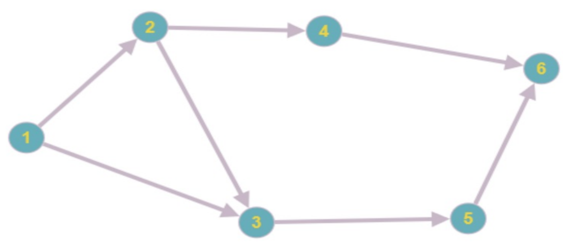
Homework Assignment 1

Instructor: Igor Shinkar *Due date: February 11, 2022*

Question 1 (3 points) *Recall the DFS based algorithm for topological sorting of directed graphs we saw in class. The algorithm runs DFS from all vertices of G, and post-numbers of the vertices in the decreasing order give a topological ordering of G.*

Prove that the pre-numbers of the vertices of G do not give a topological ordering of G (neither in increasing order nor in decreasing order). Prove it by showing a counter example for increasing order, and a counter example for decreasing order.

Question 2 (3 points) *For the graph below enumerate all its topological orderings.*



Question 3 (3 points) *Recall Kahn's algorithm for topological sorting of directed graphs we saw in class. Modify the algorithm so that*

- *If the input G is a DAG, your algorithm returns a topological sorting of G.*
- *If the input G is not a DAG, your algorithm returns a directed cycle in G.*
- *The running time on input $G = (V, E)$ is $O(|V| + |E|)$.*

Explain the running time of the algorithm, and prove its correctness.

Question 4 (3 points)

(a) *(2 points) Prove that any undirected connected graph G on n vertices has at most $n(n-1)/2$ cuts of minimum size.*

Hint: Recall Karger's randomized algorithm for min-cut we saw in class

(b) *(0.5 point) How many min-cuts are there in the cycle graph on n vertices? Explain your answer.*

(c) *(0.5 point) How many min-cuts are there in the complete graph on n vertices? Explain your answer.*

Question 5 (3 points) *Recall the randomized algorithm we saw in class, that given a bipartite graph* $G = (L, R, E)$ *with* $|L| = |R| = n$ *vertices of each side, checks if* G *has a perfect matching by looking at* $det(A^G)$*, where* $A^G_{i,j} = \begin{cases} X_{i,j}, & \text{if } (i,j) \in E \\ 0, & \text{otherwise} \end{cases}$*, with* $X_{i,j}$*'s being formal variables.*

(a) (1 points) Let $G = C_{2n}$ *be the cycle on* $2n$ *vertices* $V = \{1, 2, \ldots, 2n\}$*, and think of* C_{2n} *as a bipartite graph* $G = (L \cup R, E)$*, where* $L = \{1, 3, 5, \ldots, 2n-1\}$ *and* $R = \{2, 4, 6, 8, \ldots, 2n\}$*. Consider the matrix* $A^{C_{2n}}$*. Compute* $det(A)$ *as a formal polynomial.*

(b) (2 point) Prove that for any bipartite graph $G = (L, R, E)$ *with* $|L| = |R|$*, if* G *contains a perfect matching, the* $det(A^G)$ *is a non-zero polynomial. Prove it by assigning some real values to each variable so that the polynomial evaluates to non-zero.*

Question 6 (BONUS: 3 points) *Design a randomized algorithm that gets an undirected bipartite graph* $G = (L, R, E)$ *with* $|L| = |R|$*,, and outputs a perfect matching in* G *with probability at least* 0.99*. You may use as a subroutine the algorithm we saw in class for the decision version of this problem. You need to explain the parameters you use for the decision algorithms so that the success probability of the search algorithm is at least* 0.99*.*

Explain the running time of the algorithm, and prove its correctness.