CMPT404/705: Design and Analysis of Computing Algorithms

Homework Assignment 4

Instructor: Igor Shinkar *Due date: April 10, 2022*

Question 1 (3 points) *An r-uniform hypergraph $H = (V, E)$ is a collection of vertices V, and a collection of hyperedges E, where each hyperedge $e \in E$ is a subset of V of size $|e| = r$. The case of $r = 2$ corresponds to graphs.*

Consider the Minimum Vertex Cover problem for r-uniform hypergraphs: Given an r-uniform hypergraph the goal is to find a collection of vertices $C \subseteq V$ of minimum size such that for every $e \in E$ it holds that $e \cap C \neq \emptyset$. Design a r-approximation algorithm for the problem of finding a minimum vertex cover in an r-uniform hypergraph.

Question 2 (3 points) *Design a polynomial time algorithm that gets as input a graph G on n vertices, that is guaranteed to have a clique of size $n/\ln^5(n)$. The output of the algorithm needs to be a clique in G of size $\geq \ln(n)/\ln\ln(n)$.*

Question 3 (3 points) *Design a randomized algorithm that gets a graph $G = (V, E)$ and with probability $>$ 0.9 outputs a partition of $V = V_1 \cup V_2$ such that at most $|E|/2$ edges have both endpoints in the same V_i.*

Question 4 (3 points) *Solve the following linear program. Explain your solution.*

$$\text{maximize}_{x,y,z\in\mathbb{R}} \quad 3x + 5y - 2z$$

$$\text{subject to} \begin{cases} x & \leq 4 \\ x - y & \geq -4 \\ -x + 4y - z & \geq 3 \\ 2x - 2y + z & \geq -5 \\ x, y, z \geq 0 \end{cases}$$

Question 5 (3 points) *Recall the weighted Set Cover problem we saw in class. The input is a universe U of size n, and sets $S_1, S_2, \ldots, S_m \subseteq U$ with costs $c_i > 0$ for each set S_i, such that $\cup_{i=1}^m S_i = U$. A solution to the problem is given as a collection of sets with indices $\mathcal{I} \subseteq [m]$ such that $\cup_{i\in\mathcal{I}} S_i = U$, and its cost is $cost(\mathcal{I}) = \sum_{i\in\mathcal{I}} c_i$. An* optimal *cost for an input is given by a solution that minimizes the cost, i.e.,*

$$\mathsf{OPT} = \min_{\mathcal{I}} \left\{ \sum_{i\in\mathcal{I}} c_i : \cup_{i\in\mathcal{I}} S_i = U \right\}$$

Design a randomized polynomial time algorithm for the weighted Set-Cover problem that given a universe U of size n, sets $S_1, S_2, \ldots, S_m \subseteq U$ with costs c_i for each set S_i, and a constant $\varepsilon > 0$, outputs with probability ≥ 0.9 a solution $\mathcal{I} \subseteq [m]$ such that

$$\sum_{i\in\mathcal{I}} c_i \leq (1 + \varepsilon)\ln(n) \cdot \mathsf{OPT}.$$

You can think of ε as 0.01.

Question 6 (BONUS: 3 points - 1 point for each item) *A 3-coloring of a graph $G = (V, E)$ is a coloring the of vertices $C: V \to \{R, G, B\}$ such that no edge sees the same color on both of its endpoints; that is, for every $(u, v) \in E$ it holds that $C(v) \neq C(u)$. In the 3-coloring problem the input is a graph $G = (V, E)$, and the goal is to find a coloring the of G or report "G is not 3-colorable".*

(a) *Write an integer linear programming (ILP) formulation for the 3-coloring problem. (There is more than one way to write such an ILP. You may choose any formulation)*

(b) *Relax the ILP to LP, and show a graph that is not 3-colorable that has a feasible solution to your LP. (Note that this is a feasibility problem, and not an optimization problem)*

(c) *Write the Dual LP for the linear program in the previous item. For this you will need to add an objective function to the primal LP.*