Belief Change and Base Dependence

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Abstract

A strong intuition for AGM belief change operations, Gärdenfors suggests, is that formulas that are independent of a change should remain intact. Based on this intuition, Farias and Herzig axiomatize a *dependence* relation w.r.t. a belief set, and formalize the connection between dependence and belief change. In this paper, we introduce *base dependence* as a relation between formulas w.r.t. a *belief base*. After an axiomatization of base dependence, we formalize the connection between base dependence and a particular belief base change operation, saturated kernel contraction. Moreover, we prove that base dependence is a *reversible generalization* of Farias and Herzig's dependence. That is, in the special case when the underlying belief base is deductively closed (i.e., it is a belief set), base dependence reduces to dependence. Finally, an intriguing feature of Farias and Herzig's formalism is that it meets other criteria for dependence, namely, Keynes' conjunction criterion for dependence (CCD) and Grdenfors' conjunction criterion for independence (CCI). We show that our base dependence formalism also meets these criteria. More interestingly, we offer a more specific criterion that implies both CCD and CCI, and show our base dependence formalism also meets this new criterion.

Introduction

Belief Change

The *AGM* paradigm of *belief change* studies the dynamics of belief states in light of new information (Alchourrn, GŁrdenfors, and Makinson 1985). For theoretical simplification, AGM idealizes a belief state as a *belief set* or a *theory*: a set of logical formulas that is closed under implication.

An important variant of the original AGM approach uses *belief bases* instead of using belief sets. Belief bases are not necessarily deductively closed, and they are usually *finite*. Thus, they are more suitable to be represented in finite machines. Also, many authors have argued that, compared to belief sets, belief bases are more expressive (Hansson 2003), and they are more tolerant of inconsistency (Hansson and Wassermann 2002). Therefore, belief bases can be more useful in practice than belief sets.

Belief Change and Dependence

A long standing intuition concerning belief change is that formulas independent of a change should remain intact (GŁrdenfors 1990). In *Belief Change and Dependence*, Farias del Cerro and Herzig (1996) (FH) attempt to ground this intuition by axiomatizing a particular *dependence* relation in a close relationship to belief change. FH's work is particularly interesting and unique in the sense that it fits the original AGM model of belief change. Their stated aim is both "to give a formal account of the notion of dependence, and to employ it in belief change." This deep integration into the AGM model sets apart their work from other works on relevance or dependence in the context of belief change.

Belief Change and Base Dependence

A natural next step is to find a similar connection between dependence and *belief base contraction*. We call such a dependence (or relevance) relation *base dependence* (or *base relevance*). In this work, we provide an axiomatization of base dependence, and establish its relation to belief base contraction. Interestingly, base dependence turns out to be a *reversible generalization* of FH's dependence. That is, in the special case that a belief base is deductively closed (i.e., it is a belief set), the base dependence relation reduces to FH's original dependence relation.

Composite Dependence

One interesting aspect of FH's work is that some of the axioms that they use to capture the concept of dependence come from intuitions put forward previously. For example, Keynes (1921) holds that there is an intuitive relationship between relevance (dependence) and logical conjunction that should stay valid for any reasonable definition of relevance. Calling it the *Conjunction Criterion for Dependence*, CCD, FH formulate it as follows:

If δ depends on α and δ depends on β then δ depends on $\alpha \wedge \beta$. (CCD)

Moreover, Gärdenfors (1978) puts forward another principle that he believes should hold for relevance/dependence relations, the *Conjunction Criterion for Independence*, CCI.

If δ is independent of α and δ is independent of β then δ is independent of $\alpha \wedge \beta$.

It maintains its intuitive appeal in its contrapositive form:

If δ depends on $\alpha \wedge \beta$ then δ depends on α or δ depends on β. (CCI)

Notably, our formalism preserves both CCD and CCI. Furthermore, we offer a new and more specific criterion for dependence, which we call the *Conjunction Criterion of Dependence Factoring*, CCDF. We show that CCDF implies both CCD and CCI, and that our base dependence formalism meets the three criteria: CCDF and so CCD and CCI.

Contributions

The contributions in this work are as follows. We offer an axiomatization of base dependence for belief base formulas, and we provide characterization theorems relating base dependence to belief base contraction. We then show that base dependence is a reversible generalization of FH's dependence relation. We also show that while generalizing the dependence relation, base dependence preserves some of the most interesting properties of dependence, particularly, Keynes' CCD, and Grdenfors' CCI. Finally we put forward a more specific conjunction criterion of dependence, CCDF, that implies both CCD and CCI, and show that this new criterion is also met by base dependence.

Background

Formal Preliminaries

We assume $\mathcal{L}$ to be a propositional language defined on a finite set of propositional variables or atoms $\mathcal{V}$ with the usual Boolean operators negation $\neg$, conjunction $\wedge$, disjunction $\vee$, and implication $\rightarrow$. We will use lower case Greek letters α, β, δ, etc. as meta variables over sentences in $\mathcal{L}$. For convenience, we introduce the sentential constants $\top$ and $\bot$ representing truth and falsity respectively. A logical consequence α of a set of formulas B is represented by $B \vdash \alpha$. Also Cn is a consequence operator, a total function taking sets of formulas to sets of formulas, defined as $\mathrm{Cn}(B) = \{\alpha \mid B \vdash \alpha\}$.

For the proofs of theorems in this paper, please refer to (Oveisi 2013).

Belief Contraction

To model rational belief change, AGM uses *rationality postulates* to describe what constitute operators for belief change and it also specifies how to *construct* such operators. The belief contraction postulates are as follows:

($\div$1) $K \div \alpha$ is a belief set (closure)

($\div$2) $K \div \alpha \subseteq K$ (inclusion)

($\div$3) If $\alpha \notin K$ then $K \div \alpha = K$ (vacuity)

($\div$4) If $\nvdash \alpha$ then $\alpha \notin K \div \alpha$ (success)

($\div$5) If $\alpha \in K$ then $K \subseteq (K \div \alpha) + \alpha$ (recovery)

($\div$6) If $\vdash \alpha \leftrightarrow \beta$ then $K \div \alpha = K \div \beta$ (extensionality)

($\div$7) $K \div \alpha \cap K \div \beta \subseteq K \div \alpha \wedge \beta$ (conjunctive overlap)

($\div$8) If $\alpha \notin K \div \alpha \wedge \beta$ then $K \div \alpha \wedge \beta \subseteq K \div \alpha$ (conjunctive inclusion)

For motivation and interpretation of these postulates see (GŁrdenfors 1988). Any operator $\div$ on K satisfying postulates ($\div$1)–($\div$6) is called a basic AGM contraction operator. The *supplementary postulates* ($\div$7) and ($\div$8) specify properties of composite belief contraction operators, which involve contraction by conjunction of sentences. Indeed, AGM also provides a third composite contraction postulate:

Either $K \div \alpha \wedge \beta = K \div \alpha$, or
$K \div \alpha \wedge \beta = K \div \beta$, or (conjunctive factoring)
$K \div \alpha \wedge \beta = K \div \alpha \cap K \div \beta$.

A basic AGM contraction operator that satisfies conjunctive factoring, also satisfies both ($\div$7) and ($\div$8), and vice versa (Alchourrn, GŁrdenfors, and Makinson 1985).

Some of our beliefs are more *epistemically entrenched* than others, making them harder to give up. Based on this intuition, Grdenfors (1988) introduced epistemic entrenchment, and defined the properties of an order relation $\leq$ between sentences. Grdenfors and Makinson (1988) show that an AGM contraction operator $\div$ can be constructed using a $\leq$ relation, and that, conversely, an epistemic entrenchment relation $\leq$ can be constructed using an AGM contraction operator $\div$.

Turning now to belief base contraction, as with Hansson (1999), we assume that an operation needs to at least satisfy success and inclusion to be considered a belief contraction.

Definition 1. An operator $\div$ for a set B is an operator of contraction if and only if it satisfies success and inclusion.

Some other contraction axioms are listed below:

$B \cap \mathrm{Cn}(B \div \alpha) \subseteq B \div \alpha$ (relative closure)

If $\alpha \in \mathrm{Cn}(B')$ iff $\beta \in \mathrm{Cn}(B')$ for all $B' \subseteq B$ then $B \div \alpha = B \div \beta$ (uniformity)

If $\beta \in B$ and $\beta \notin B \div \alpha$ then $\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ for some $B \div \alpha \subseteq B' \subseteq B$ (relevance)

If $\beta \in B$ and $\beta \notin B \div \alpha$ then $\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ for some $B' \subseteq B$ (core-retainment)

Different combinations of the contraction axioms specify different contraction operations. In particular, we note the following:

Partial meet contraction: $(\div 1)$–$(\div 6)$ (Alchourrn, GŁrdenfors, and Makinson 1985); or success, inclusion, relevance and uniformity (Hansson 2003).

Kernel contraction: success, inclusion, core-retainment and uniformity (Hansson 1995).

Saturated (or smooth) kernel contraction: success, inclusion, core-retainment, uniformity and relative closure (Hansson 1995).

Belief Change and Dependence

Farias del Cerro and Herzig (1996) formalized the notion of dependence and its connection with belief change, in a similar approach to Gärdenfors and Makinson (1988) for epistemic entrenchment. To formalize dependence, they investigate a binary relation $\leadsto$ on formulas. $\alpha \leadsto \beta$ reads as "β depends on α" (or equivalently "α is relevant to β"). Independence, then, is denoted by $\not\leadsto$, which is the complement of $\leadsto$, so $\alpha \not\leadsto \beta$ reads as "β is independent of α" (or "α is irrelevant to β"). They provide the following axiomatization of dependence.

If $\alpha \leftrightarrow \beta$ and $\alpha \leadsto \delta$ then $\beta \leadsto \delta$ $\quad$ (LE^l)

If $\alpha \leftrightarrow \beta$ and $\delta \leadsto \alpha$ then $\delta \leadsto \beta$ $\quad$ (LE^r)

If $\alpha \wedge \beta \leadsto \delta$ then $\alpha \leadsto \delta$ or $\beta \leadsto \delta$ $\quad$ (CCI^l)

If $\delta \leadsto \alpha \wedge \beta$ then $\delta \leadsto \alpha$ or $\delta \leadsto \beta$ $\quad$ (CCI^r)

$\alpha \in K$ iff either $\vdash \alpha$ or $\alpha \leadsto \beta$ for some β $\quad$ (Def-K)

If $\alpha \leadsto \beta$ then $\alpha \leadsto \alpha$ $\quad$ (Cond-ID)

If $\vdash \alpha \vee \beta$ then $\alpha \not\leadsto \beta$ $\quad$ (Disj)

If $\alpha \leadsto \delta$ and $\alpha \wedge \beta \leadsto \alpha$ then $\alpha \wedge \beta \leadsto \delta$ $\quad$ (CCD_0^l)

If $\delta \leadsto \alpha$ and $\beta \leadsto \beta$ then $\delta \leadsto \alpha \wedge \beta$ $\quad$ (CCD_0^r)

The following are also derivable principles:

If $\alpha \leadsto \delta$ and $\beta \leadsto \delta$ then $\alpha \wedge \beta \leadsto \delta$. $\quad$ (CCD^l)

If $\delta \leadsto \alpha$ and $\delta \leadsto \beta$ then $\delta \leadsto \alpha \wedge \beta$. $\quad$ (CCD^r)

For motivation and interpretation of these postulates, please refer to their work. Here, it suffices to note that Keynes' CCD and Grdenfors' CCI are represented by CCD^l and CCI^l, respectively.

FH's dependence relation is defined as follows:

Definition 2. A relation $\leadsto$ is a *dependence* relation if and only if it satisfies the axioms LE^l, LE^r, CCI^l, CCI^r, Def-K, Cond-ID, Disj, CCD_0^l and CCD_0^r.

For studying the relationship between dependence and belief change, FH use the following *preservation criterion* as their guiding principle:

> "If a belief state is revised by a sentence A, then all sentences in K that are independent of the validity of A should be retained in the revised state of belief" (GŁrdenfors 1990).

For example, if $\beta \in K$ to begin with, but $\beta \notin K \div \alpha$, then we can say that β depends on α, or $\alpha \leadsto \beta$.

Similar to epistemic entrenchment, to provide the connection between dependence and contraction, FH introduce two conditions, namely, Cond$\leadsto$ and Cond$\div$.

$$\alpha \leadsto \beta \text{ iff } \beta \in K \text{ and } \beta \notin K \div \alpha. \qquad (\mathsf{Cond}\leadsto)$$

This equivalence condition allows one to define $\leadsto$ based on a given AGM contraction operation $\div$ for belief set K.

Theorem 3 (FH). *Given two relations $\leadsto$ and $\div$ such that* Cond$\leadsto$ *holds, if $\div$ is an AGM contraction, then $\leadsto$ is a dependence relation.*

The next condition allows defining an AGM contraction operation $\div$, given a dependence relation $\leadsto$.

$$\beta \in K \div \alpha \text{ iff either } \vdash \beta \text{ or } \beta \leadsto \beta \text{ and } \alpha \not\leadsto \beta. \qquad (\mathsf{Cond}\div)$$

An AGM contraction operation $\div$ is defined with respect to some belief set K. Thus, $\div$ obtained via Cond$\div$ also requires an associated K to be specified. For this purpose, FH provide the following definition for the belief set $K_\leadsto$:

$$K_\leadsto = \{\alpha \mid \vdash \alpha \text{ or } \alpha \leadsto \beta \text{ for some } \beta\}.$$

For the sake of brevity, FH simply use K to refer to $K_\leadsto$ afterwards. The next theorem defines a contraction operation, given a dependence relation using Cond$\div$.

Theorem 4 (FH). *Given two relations $\leadsto$ and $\div$ such that* Cond$\div$ *holds, if $\leadsto$ is a dependence relation then $\div$ is an AGM contraction.*

As the last step, FH need to complete the link between AGM contraction and dependence using a characterization theorem to state that for any two arbitrary relations $\leadsto$ and $\div$ that satisfy Cond$\leadsto$, $\div$ is an AGM contraction iff $\leadsto$ is a dependence relation. To achieve this, however, it turns out that they first have to make the following assumption:

Remark 5. In order to establish an axiomatic characterization based on Cond$\leadsto$, it is assumed that the relation $\div$ satisfies inclusion, $K \div \alpha \subseteq K$.

Theorem 6 (FH). *Let two relations $\leadsto$ and $\div$ be such that $\div$ satisfies* inclusion *and that* Cond$\leadsto$ *holds. Then $\div$ is an AGM contraction if and only if $\leadsto$ is a dependence relation.*

This completes FH's work. Also another characterization theorem that they provide for a weaker dependence relation proves to be useful in our study.

Theorem 7 (FH). *Let two relations $\leadsto$ and $\div$ be such that $\div$ satisfies* inclusion *and that* Cond$\leadsto$ *holds. Then $\div$ is a basic AGM contraction satisfying $(\div 1)$–$(\div 6)$ if and only if $\leadsto$ is a dependence relation satisfying* LE^l, LE^r, CCI^r, Def-K, Cond-ID, Disj *and* CCD_0^r.

Belief Change and Base Dependence

Overview and Problem Definition

Gärdenfors' preservation criterion, as with FH, lays the foundation of the present work. Our work is another attempt to connect notions of dependence and belief change, but using belief bases instead of belief sets.

As discussed, belief bases are a generalization of belief sets. Hence, it seems natural to anticipate that base dependence also be a generalization of FH's dependence relation. Furthermore, base dependence should ideally be a *reversible generalization* of dependence. That is, where a base dependence relation corresponds to a belief set, the base dependence relation should reduce to FH's dependence.

Finally, one significant feature of FH's formalism is that it adheres to Keynes' CCD and Gärdenfors' CCI. We would like to preserve this characteristic of FH's dependence while generalizing it to base dependence. It turns out that our base dependence formalism meets a third maxim, CCDF, that implies both CCD and CCI.

Characteristics of an Anticipated Solution

A formalization of Gärdenfors' preservation criterion that involves belief bases instead of belief sets should still retain the general scheme of FH's work. More specifically, we need an *axiomatization of base dependence*, as well as a suitable corresponding *belief base contraction*. We also need a *conditional, similar to FH's* Cond$\rightsquigarrow$, that allows construction of a base dependence given a base contraction, and another *conditional, similar to FH's* Cond$\div$, that allows construction of a base contraction using a base dependence.

As discussed, base dependence should be a reversible generalization of FH's dependence. Naturally, an appropriate base contraction to correspond to base dependence must also be a *reversible generalization of AGM contraction*.

Finally, our base dependence and its corresponding base contraction must be such that CCD and CCI (and ideally CCDF) are satisfied.

These characteristics are pictured in Figure 1, located toward the end of our paper. The remainder of this work is to establish the background concepts and theorems necessary to justify the diagram in Figure 1.

Base Dependence

The meaning of "dependence" in base dependence is the same as what Fariñas and Herzig (and Grdenfors) studied, which refers to the dependence or relevance of logical statements towards one another. Using their notation, we read $\alpha \rightsquigarrow \beta$ as "β depends on α" or "doubting in α leads to doubting in β."

In FH's study, dependence can only happen between (contingent) sentences from K:

$$\text{If } \alpha \rightsquigarrow \beta \text{ then } \alpha \in K \text{ and } \beta \in K.$$

If B is a base for K, $K = \mathrm{Cn}(B)$, then we have:

$$\text{If } \alpha \rightsquigarrow \beta \text{ then } \alpha \in \mathrm{Cn}(B) \text{ and } \beta \in \mathrm{Cn}(B). \quad (1)$$

One way to generalize the dependence relation $\rightsquigarrow$ is to make α or β be from B instead of $\mathrm{Cn}(B)$. Therefore, our sought-for *base dependence* should somehow involve formulas explicitly mentioned in a belief base, thus the name.

Using $\bar{\rightsquigarrow}$ to denote base dependence, we read $\alpha \bar{\rightsquigarrow} \beta$ as "β *base-depends* on α," which is the same as $\alpha \rightsquigarrow \beta$ except that $\alpha \bar{\rightsquigarrow} \beta$ also implies that α or β or both are formulas in the base. Now, we need to decide which one of these three alternatives should be the case.

$$\text{If } \alpha \bar{\rightsquigarrow} \beta \text{ then } \alpha \in B \text{ and } \beta \in \mathrm{Cn}(B) \quad (2a)$$

$$\text{If } \alpha \bar{\rightsquigarrow} \beta \text{ then } \alpha \in \mathrm{Cn}(B) \text{ and } \beta \in B \quad (2b)$$

$$\text{If } \alpha \bar{\rightsquigarrow} \beta \text{ then } \alpha \in B \text{ and } \beta \in B. \quad (2c)$$

We believe the third alternative (2c), requiring both α and β to be in B, appears to be too strong to try first. The second alternative (2b) offers a more interesting semantics, compared to the first alternative (2a). The second alternative means that "doubting in α leads to doubting in β from the base." It allows us to study how the statements in the *base* depend on, or are susceptible to, changes of other statements. Stated in terms of belief change, it means contracting B by any formula α from the *infinite* set $\mathrm{Cn}(B)$ can result in removal of β from the *finite* set $B \setminus B \div \alpha$. On the other hand, the first alternative (2a) means contracting B by any formula α from the usually *finite* set B can result in removal of β from the *infinite* set $\mathrm{Cn}(B) \setminus \mathrm{Cn}(B \div \alpha)$.

Therefore, although the alternatives (2a) and (2c) remain open and may be found useful in other studies, we proceed with (2b), and from now on we assume $\alpha \bar{\rightsquigarrow} \beta$ requires that $\beta \in B$:

$$\text{If } \alpha \bar{\rightsquigarrow} \beta \text{ then } \beta \in B. \quad (3)$$

It turns out that (3) does not need to be explicitly specified as an axiom. Rather, it will be implied by other axioms and conditions for base dependence which will be put forward in the upcoming sections (e.g., Def-B and Cond$\bar{\rightsquigarrow}$).

Example 8. Assume Mary believes that p, q and $q \rightarrow r$; i.e., $B = \{p, q, q \rightarrow r\}$. By implication, she also believes that r, as it is entailed by q and $q \rightarrow r$. Now, say, for some reason, she starts to doubt that r is true. Consequently, this leads her to also doubt either q, or $q \rightarrow r$, or even both. Thus we know that at least one of $r \bar{\rightsquigarrow} q$ or $r \bar{\rightsquigarrow} (q \rightarrow r)$ hold, and that $r \not\bar{\rightsquigarrow} p$ and $r \not\bar{\rightsquigarrow} r$. □

Mutual Construction

From Contraction to Base Dependence As discussed earlier, FH use Cond$\rightsquigarrow$ to construct a dependence relation via a given AGM contraction. Cond$\rightsquigarrow$ can be straightforwardly transformed to an equivalent base-generated representation. That is, if $K = \mathrm{Cn}(B)$ and $\div$ is a base-generated contraction, which exists for any given AGM contraction (Hansson 1993), then Cond$\rightsquigarrow$ can be stated as follows:

$$\alpha \rightsquigarrow \beta \;\text{ iff }\; \beta \in \mathrm{Cn}(B) \text{ and } \beta \notin \mathrm{Cn}(B \div \alpha). \quad (\text{Cond}\rightsquigarrow)$$

Note that we have used the same name "Cond$\rightsquigarrow$" here since this is only an alternative representation. This new, base-generated representation makes it easier to compare and contrast Cond$\rightsquigarrow$ with the corresponding conditions for belief bases that will be introduced shortly.

If $\alpha \rightsquigarrow \beta$ then β is retracted as a result of α's contraction, so $\beta \in [\mathrm{Cn}(B) \setminus \mathrm{Cn}(B \div \alpha)]$. For base dependence, we assume that (3) holds: if $\alpha \bar{\rightsquigarrow} \beta$ then $\beta \in B$. Thus, it is intuitively appealing to say that if $\alpha \bar{\rightsquigarrow} \beta$, then β is retracted from B as a result of α's contraction, so $\beta \in [B \setminus B \div \alpha]$. With this intuition in mind, we propose the following to correspond to FH's Cond$\rightsquigarrow$:

$$\alpha \bar{\rightsquigarrow} \beta \;\text{ iff }\; \beta \in B \text{ and } \beta \notin B \div \alpha. \quad (\text{Cond}\bar{\rightsquigarrow})$$

Notice that in common with FH, if α is a tautology, no formula contributes to its truth, and $\alpha \bar{\rightsquigarrow} \beta$ cannot hold for any β. Also, if β is not in B, then $\alpha \bar{\rightsquigarrow} \beta$ cannot hold for any α by definition. Therefore, both $\beta \rightsquigarrow \beta$ and $\beta \bar{\rightsquigarrow} \beta$ imply that β is contingent, and the latter additionally implies that β is in B. This will prove useful in the next section when constructing contraction operations using (base) dependence, via Cond$\div$ and Cond$\bar{\div}$.

From Base Dependence to Contraction We first provide a simplifying notation $\vdash_B$ to help represent *tautologies present in the base*.

Definition 9. Given a base B and an entailment relation $\vdash$, the *base entailment* relation $\vdash_B$ is defined as follows: $A \vdash_B \beta$ if and only if $\beta \in B$ and $A \vdash \beta$.

A useful special case is when $A = \emptyset$. For example, $\vdash_B \beta$ means β is a tautology in the base: $\beta \in B$ and $\vdash \beta$. One important usage is to help handling tautologies in base dependence axioms. Such axioms are primarily concerned with contingencies, but they have to also deal with tautologies, usually as exceptional cases.

Next, we need to reconstruct belief bases, given base dependence relations. We saw that FH provide the following:

$$K_{\rightsquigarrow} = \{\alpha \mid \vdash \alpha \text{ or } \alpha \rightsquigarrow \beta \text{ for some } \beta\}.$$

Note that we could swap the role of α and β above and obtain the same results:

$$K_{\rightsquigarrow} = \{\beta \mid \vdash \beta \text{ or } \alpha \rightsquigarrow \beta \text{ for some } \alpha\}.$$

Similarly, a base dependence relation $\bar{\rightsquigarrow}$ is associated with a belief base B. Thus, it should be possible to recreate the associated belief base B via a given $\bar{\rightsquigarrow}$ relation:

$$B_{\bar{\rightsquigarrow}} = \{\beta \mid \alpha \mathrel{\bar{\rightsquigarrow}} \beta \text{ for some } \alpha\}.$$

Note, however, that one caveat is that $B_{\bar{\rightsquigarrow}}$ will not contain any *tautologies* that may be in B. Still, we can say that B and $B_{\bar{\rightsquigarrow}}$ are equivalent for most practical purposes. Also, their closure is obviously equivalent:

$$\mathrm{Cn}(B_{\bar{\rightsquigarrow}}) = \mathrm{Cn}(B).$$

If in addition to the base dependence relation $\bar{\rightsquigarrow}$, we are also given $\vdash_B$ that identifies tautologies in the base, then we can have the following, which guarantees that $B_{\bar{\rightsquigarrow}} = B$:

$$B_{\bar{\rightsquigarrow}} = \{\beta \mid \vdash_B \beta \text{ or } \alpha \mathrel{\bar{\rightsquigarrow}} \beta \text{ for some } \alpha\}$$

In the rest of this work, we assume $B_{\bar{\rightsquigarrow}} = B$. In the worst case scenario, there are tautologies in B and only $\bar{\rightsquigarrow}$ is given, so the tautologies in B are not present in $B_{\bar{\rightsquigarrow}}$.

Now for Cond$\bar{\div}$, again we start with Cond$\div$. As in the case of Cond$\rightsquigarrow$, we present a straightforward transformation to the equivalent base-generated operation. Again, we reuse the equation's name, "Cond$\div$":

$$\beta \in \mathrm{Cn}(B \div \alpha) \text{ iff either } \vdash \beta \text{ or } \beta \rightsquigarrow \beta \text{ and } \alpha \not\rightsquigarrow \beta. \tag{Cond$\div$}$$

Cond$\div$ says $\beta \in \mathrm{Cn}(B \div \alpha)$ means either that β is a tautology, or that β is a contingent truth, $\beta \rightsquigarrow \beta$, but contraction by α does not lead to retraction of β, meaning $\alpha \not\rightsquigarrow \beta$.

To adapt this for belief bases, we need something along the line of the following: $\beta \in B \div \alpha$ means either that β is a tautology in B, $\vdash_B \beta$, or that β is a contingent truth in B, $\beta \mathrel{\bar{\rightsquigarrow}} \beta$, but contraction by α does not lead to retraction of β from B, $\alpha \mathrel{\bar{\not\rightsquigarrow}} \beta$:

$$\beta \in B \div \alpha \text{ iff either } \vdash_B \beta \text{ or } \beta \mathrel{\bar{\rightsquigarrow}} \beta \text{ and } \alpha \mathrel{\bar{\not\rightsquigarrow}} \beta. \tag{Cond$\bar{\div}$}$$

Basic Postulates of Base Dependence

A goal of this work is to provide an axiomatization of base dependence as a generalization of FH's dependence. It turns out that some of base dependence axioms closely resemble dependence axioms (e.g. Cond-ID^B), and some remain valid and derivable but are no longer needed as axioms (e.g. Disj^B). Yet, there are also some other axioms offered for base dependence (e.g. redundancy) that are not similar to any of the dependence axioms. The following are the basic postulates of base dependence.

$$\beta \in B \text{ iff either } \vdash_B \beta \text{ or } \alpha \mathrel{\bar{\rightsquigarrow}} \beta \text{ for some } \alpha. \tag{Def-B}$$

$$\text{If } \alpha \mathrel{\bar{\rightsquigarrow}} \beta \text{ then } \beta \mathrel{\bar{\rightsquigarrow}} \beta. \tag{Cond-ID^B}$$

$$\text{If } \alpha \in \mathrm{Cn}(B') \text{ iff } \beta \in \mathrm{Cn}(B') \text{ for all } B' \subseteq B \text{ then } \alpha \mathrel{\bar{\rightsquigarrow}} \delta \text{ iff } \beta \mathrel{\bar{\rightsquigarrow}} \delta. \tag{conjugation}$$

$$\text{If } \alpha \mathrel{\bar{\rightsquigarrow}} \beta \text{ then } \alpha \notin \mathrm{Cn}(B') \text{ and } \alpha \in \mathrm{Cn}(B' \cup \{\beta\}) \text{ for some } B' \subseteq B. \tag{contribution}$$

$$\text{If } \alpha \in \mathrm{Cn}(B') \text{ and } B' \subseteq B \text{ then either } \vdash \alpha \text{ or } \alpha \mathrel{\bar{\rightsquigarrow}} \beta \text{ for some } \beta \in B'. \tag{modularity}$$

$$\text{If } \beta \in \mathrm{Cn}(B') \text{ and } B' \subseteq B \text{ then either } \alpha \mathrel{\bar{\not\rightsquigarrow}} \beta \text{ or } \alpha \mathrel{\bar{\rightsquigarrow}} \delta \text{ for some } \delta \in B'. \tag{redundancy}$$

A brief description of these axioms is as follows:

(Def-B) To say that β is in the base, $\beta \in B$, is equivalent to saying either that β is a tautology in the base, $\vdash_B \beta$, or that it is a contingency in the base. The contingent truth of β then has to be inferentially relevant to some (contingent) formula α (where α could be β), so $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$.

(Cond-ID^B) The inferential relevance between α and β means that neither is a tautology. Also, β is in the base because $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$. Thus, β is a contingency in the base, or $\beta \mathrel{\bar{\rightsquigarrow}} \beta$.

(conjugation) When α and β are logically equivalent, $\vdash \alpha \leftrightarrow \beta$, there is inferential relevance between α and δ from B if and only if there is inferential relevance between β and δ. This makes sense not only for α and β, but also for the formulas that are true *just because* α or β are true.

(contribution) This axiom says that if β from B is inferentially relevant to α then β must somehow contribute to the justification of α.

(modularity) Consider a subset $B' \subseteq B$ that implies α, $\alpha \in \mathrm{Cn}(B')$. This could be because α is a tautology. But when $\not\vdash \alpha$, there is some β from the same subset B' that base-depends on α.

(redundancy) To consider the principal case of redundancy, assume that: $\beta \in \mathrm{Cn}(B')$ and $B' \subseteq B$ and $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$. When $\beta \in B'$, $\alpha \mathrel{\bar{\rightsquigarrow}} \delta$ for some $\delta \in B'$ trivially holds because then δ could be β which means $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$, which is assumed. When $\beta \notin B'$, there is some redundancy in the base B because on the one hand $\beta \in B$ (as $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$), and on the other hand there is $B' \subseteq B$ such that $\beta \notin B'$ but $\beta \in \mathrm{Cn}(B')$. Thus, in order for $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ to hold, $\alpha \mathrel{\bar{\rightsquigarrow}} \delta$ should also hold at least for one formula $\delta \in B'$.

The following principles, which are derivable from the above axioms, also have counterpart in FH's framework:

If $\vdash \alpha \vee \beta$ then $\alpha \not\rightsquigarrow \beta$. (Disj^B)

If $\vdash \alpha \leftrightarrow \beta$ and $\alpha \rightsquigarrow \delta$ then $\beta \rightsquigarrow \delta$. (LE^B)

Example 10. Assume that $B = \{p \leftrightarrow q, p \vee q, p\}$, and that $\rightsquigarrow$ is a relation such that $(p \wedge q) \rightsquigarrow (p \vee q)$ and $(p \wedge q) \not\rightsquigarrow p$ hold. We show that $\rightsquigarrow$ violates redundancy. Let $B' = \{p\}$. Clearly $B' \subseteq B$ and $p \vee q \in \mathrm{Cn}(B')$, and, by assumption, $(p \wedge q) \rightsquigarrow (p \vee q)$. Thus, by redundancy, $(p \wedge q) \rightsquigarrow \delta$ for some $\delta \in B'$. Since $B' = \{p\}$, δ has to be p, so $(p \wedge q) \rightsquigarrow p$. This contradicts the initial assumption that $(p \wedge q) \not\rightsquigarrow p$. □

Definition 11. A relation $\rightsquigarrow$ is a *base dependence* if and only if it satisfies the axioms Def-B, Cond-ID^B, conjugation, contribution, modularity and redundancy.

Notice that so far we have not specified any criteria on how to handle conjunctions, which we will do shortly.

Saturated Kernel Contraction and Base Dependence

As discussed, we need a base contraction to correspond to base dependence. We also saw that a suitable candidate base contraction must be a reversible generalization of AGM contraction. Indeed, saturated kernel contraction is a subclass of kernel contraction that is a reversible generalization of basic AGM contraction. That is, saturated kernel contraction is a base contraction, and when it corresponds to a belief set, it is equivalent to AGM partial meet contraction (Hansson 1995).

In this section, we show that indeed saturated kernel contraction and base dependence have a mutual correspondence.

From Base Dependence to Contraction To construct a contraction operator $\div$, assume all the following are present: a base dependence relation $\rightsquigarrow$ (Definition 11), a list of tautologies present in the base $T \subseteq B$ where $T = \{\beta \mid \vdash_B \beta\}$, and the Cond$\div$.

We do *not* need to assume that B is provided because it can be obtained using $\rightsquigarrow$ and $\vdash_B$.

Theorem 12 states that the contraction operator $\div$ obtained from $\rightsquigarrow$ is indeed a saturated kernel contraction.

Theorem 12 (Base Dependence to Contraction). *Given relations $\rightsquigarrow$ and $\div$ for base B such that* Cond$\div$ *holds, if $\rightsquigarrow$ is a base dependence, then $\div$ is a saturated kernel contraction.*

To prove this theorem, we assume the Cond$\div$ and postulates of base dependence hold, and we show one by one that the postulates of saturated kernel contraction also hold.

From Contraction to Base Dependence This subsection shows how to obtain a base dependence $\rightsquigarrow$ relation given a saturated kernel contraction operator $\div$. We assume the following are present: a saturated kernel contraction operator $\div$, and the Cond$\rightsquigarrow$. Theorem 13 states that, given the above assumptions, all axioms of base dependence $\rightsquigarrow$ relation are satisfied.

Theorem 13 (Contraction to Base Dependence). *Given relations $\rightsquigarrow$ and $\div$ for base B such that* Cond$\rightsquigarrow$ *holds, if $\div$ is a saturated kernel contraction, then $\rightsquigarrow$ is a base dependence.*

For this theorem, assuming the Cond$\rightsquigarrow$ and axioms of saturated kernel contraction, we show one by one that the properties of base dependence hold.

Axiomatic Characterization We now need an axiomatic characterization theorem. We also adopt the FH assumption in Remark 5, which means that inclusion needs to be assumed in the characterization theorem. The rationale for this assumption is as follows. When constructing the $\rightsquigarrow$ relation using a contraction operation via Cond$\rightsquigarrow$, the set of all β such that $\alpha \rightsquigarrow \beta$ is equal to those $\beta \in B$ and $\beta \notin B \div \alpha$, or using set difference notation $\beta \in B \setminus (B \div \alpha)$. We know as a matter of fact that $B \div \alpha \subseteq B$ holds because, by Definition 1, any contraction operator satisfies inclusion. However, even if, for the sake of argument, $\div$ did not satisfy inclusion and there were some statements in $B \div \alpha$ that were not in B, such statements would have been lost in the set difference $\beta \in B \setminus (B \div \alpha)$. That, in turn, means that to use $\rightsquigarrow$ to construct a contraction $\div$ via Cond$\rightsquigarrow$, we do not have enough information to prove or disprove inclusion. Instead, we have to assume that $\div$ already satisfies inclusion. Since all contraction operations satisfy inclusion, this assumption is not a serious loss of generality.

Theorem 14 (Characterization). *Let the relations $\rightsquigarrow$ and $\div$ for base B be such that $\div$ satisfies* inclusion *and that* Cond$\rightsquigarrow$ *holds. Then, $\div$* is a saturated kernel contraction if and only if $\rightsquigarrow$ is a base dependence.

To prove this characterization theorem, we show that in presence of inclusion, Cond$\rightsquigarrow$ entails Cond$\div$. Thus, assuming inclusion and Cond$\rightsquigarrow$, based on Theorems 12 and 13, saturated kernel contraction and base dependence are logically equivalent.

Conjunction Criterion of Dependence Factoring

We have seen Keynes' CCD and Grdenfors' CCI, and the respective dependence axioms CCD^l and CCI^l. Likewise, we have:

If $\alpha \rightsquigarrow \delta$ and $\beta \rightsquigarrow \delta$ then $\alpha \wedge \beta \rightsquigarrow \delta$. (CCD^B)

If $\alpha \wedge \beta \rightsquigarrow \delta$ then $\alpha \rightsquigarrow \delta$ or $\beta \rightsquigarrow \delta$. (CCI^B)

CCD and CCI state intuitions regarding dependence on conjunctions in the form of *conditional* statements. One wonders whether it is possible to capture such intuitions regarding dependence on conjunctions using *equivalences*. Such a statement would have to capture different cases. That is, for any reasonable dependence relation, at least one of the following statements hold:

Case 1: The set of formulas that depend on $\alpha \wedge \beta$ is *the same as* the set of those that depend on α

Case 2: The set of formulas that depend on $\alpha \wedge \beta$ is *the same as* the set of those that depend on β

Case 3: The set of formulas that depend on $\alpha \wedge \beta$ is *the same as* the set of those that depend on α *or* depend on β

Using set notation, these cases can be stated as follows:

$$\begin{aligned} \text{Either } & \{\delta \mid \alpha \wedge \beta \rightsquigarrow \delta\} = \{\delta \mid \alpha \rightsquigarrow \delta\}, \text{ or} \\ & \{\delta \mid \alpha \wedge \beta \rightsquigarrow \delta\} = \{\delta \mid \beta \rightsquigarrow \delta\}, \text{ or} \\ & \{\delta \mid \alpha \wedge \beta \rightsquigarrow \delta\} = \{\delta \mid \alpha \rightsquigarrow \delta\} \cup \{\delta \mid \beta \rightsquigarrow \delta\} \end{aligned} \tag{4}$$

or equivalently,

$$\begin{aligned}\text{Either } &[\alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta_1 \text{ iff } \alpha \mathrel{\bar{\rightsquigarrow}} \delta_1], \text{ or}\\ &[\alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta_2 \text{ iff } \beta \mathrel{\bar{\rightsquigarrow}} \delta_2], \text{ or} \qquad (\mathsf{CCDF}^B)\\ &[\alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta_3 \text{ iff } \alpha \mathrel{\bar{\rightsquigarrow}} \delta_3 \text{ or } \beta \mathrel{\bar{\rightsquigarrow}} \delta_3].\end{aligned}$$

Each line of CCDF^B needs to use a unique variable name δ_1, δ_2 and δ_3 because, in each line of (4), $\{\delta \mid \alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta\}$ refers to a different set.

CCDF^B is a formalization of the intuition expressed in the three cases above, which we restate more concisely as follows, calling it the *Conjunction Criterion of Dependence Factoring*, CCDF:

> The set of all formulas that depend on $\alpha \wedge \beta$ is the same as the set of all formulas that depend on α, or on β, or on either of them. (CCDF)

Indeed, CCDF may be considered as a third maxim for dependence of conjunctions in addition to Keynes' CCD and Grdenfors' CCI.

As a side note, although it seems that the third clause of CCDF^B should be redundant, in light of the first two, in fact it isn't.

Example 15. Assume α, β, θ_1, θ_2 and θ_3 are formulas and $\bar{\rightsquigarrow}$ is a relation such that

$$\begin{array}{lll}\alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \theta_1 & \alpha \mathrel{\bar{\rightsquigarrow}} \theta_1 & \beta \mathrel{\not\bar{\rightsquigarrow}} \theta_1\\ \alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \theta_2 & \alpha \mathrel{\not\bar{\rightsquigarrow}} \theta_2 & \beta \mathrel{\bar{\rightsquigarrow}} \theta_2\\ \alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \theta_3 & \alpha \mathrel{\bar{\rightsquigarrow}} \theta_3 & \beta \mathrel{\bar{\rightsquigarrow}} \theta_3.\end{array}$$

Clearly, $\bar{\rightsquigarrow}$ violates the first two clauses of CCDF^B, but not the third one. This may be easier to see using (4). Note that $\{\delta \mid \alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta\} = \{\theta_1, \theta_2, \theta_3\}$, $\{\delta \mid \alpha \mathrel{\bar{\rightsquigarrow}} \delta\} = \{\theta_1, \theta_3\}$ and $\{\delta \mid \beta \mathrel{\bar{\rightsquigarrow}} \delta\} = \{\theta_2, \theta_3\}$, which satisfy the third clause of (4) but not the first two. □

As a second note, CCDF^B is only one way of formalizing CCDF, using base dependence relation; of course, it can also be formalized using FH's dependence relation as shown below, which we call CCDF^l:

$$\begin{aligned}\text{Either } &[\alpha \wedge \beta \rightsquigarrow \delta_1 \text{ iff } \alpha \rightsquigarrow \delta_1], \text{ or}\\ &[\alpha \wedge \beta \rightsquigarrow \delta_2 \text{ iff } \beta \rightsquigarrow \delta_2], \text{ or} \qquad (\mathsf{CCDF}^l)\\ &[\alpha \wedge \beta \rightsquigarrow \delta_3 \text{ iff } \alpha \rightsquigarrow \delta_3 \text{ or } \beta \rightsquigarrow \delta_3].\end{aligned}$$

Finally, an important observation here is that CCDF^B is a more specific criterion than CCD^B and CCI^B, and it implies both of them.

Theorem 16. *If a relation* $\bar{\rightsquigarrow}$ *satisfies* CCDF^B*, then it also satisfies both* CCD^B *and* CCI^B.

Notice that although Theorem 16 is stated in terms of base dependence $\bar{\rightsquigarrow}$, it does not have to be. Indeed, the theorem (and its proof) may straightforwardly be restated in terms of CCDF that implies both CCD and CCI. As such, the dependence version of CCDF, i.e. CCDF^l, also implies both FH's CCD^l and CCI^l.

The following are all the conjunction criteria and related axioms we have discussed:

Criterion	Dependence	Base Dependence
CCD (Keynes)	CCD^l (FH)	CCD^B
CCI (Grdenfors)	CCI^l (FH)	CCI^B
CCDF	CCDF^l	CCDF^B

We may now state axiomatic characterization and its associated theorems, with the addition of corresponding conjunction criteria as follows.

Theorem 17 (Base Dependence to Contraction). *Given relations* $\bar{\rightsquigarrow}$ *and* $\div$ *for base* B *such that* Cond$\bar{\div}$ *holds, if* $\bar{\rightsquigarrow}$ *is a base dependence that satisfies* CCDF^B*, then* $\div$ *is a saturated kernel contraction that satisfies* conjunctive factoring.

Theorem 18 (Contraction to Base Dependence). *Given relations* $\bar{\rightsquigarrow}$ *and* $\div$ *for base* B *such that* Cond$\bar{\rightsquigarrow}$ *holds, if* $\div$ *is a saturated kernel contraction that satisfies* conjunctive factoring*, then* $\bar{\rightsquigarrow}$ *is a base dependence that satisfies* CCDF^B.

Theorem 19 (Main Characterization). *Let the relations* $\bar{\rightsquigarrow}$ *and* $\div$ *for base* B *be such that* $\div$ *satisfies* inclusion, $B \div \alpha \subseteq B$*, and that* Cond$\bar{\rightsquigarrow}$ *holds:* $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$*. Then,* $\div$ *is a saturated kernel contraction that satisfies* conjunctive factoring *if and only if* $\bar{\rightsquigarrow}$ *is a base dependence that satisfies* CCDF^B.

Base Dependence as a Reversible Generalization of Dependence

Last, we show that base dependence is a reversible generalization of FH's dependence.

Theorem 20 (Dependence Generalization). *Let relations* $\bar{\rightsquigarrow}$, $\rightsquigarrow$ *and* $\div$ *for base* B *be such that* Cond$\bar{\rightsquigarrow}$ *and* Cond$\rightsquigarrow$ *hold and* inclusion *is satisfied. In the case where* B *is logically closed,*

(1) *the following are logically equivalent:*
- a) $\bar{\rightsquigarrow}$ *is a base dependence, which satisfies* Def-B, Cond-ID^B, conjugation, contribution, modularity *and* redundancy
- b) $\rightsquigarrow$ *is a dependence that satisfies* Def-K, Cond-ID, Disj, LE^l, LE^r, CCI^r *and* CCD^r_0
- c) $\div$ *is a saturated kernel contraction, which satisfies* success, inclusion, core-retainment, uniformity *and* relative closure
- d) $\div$ *is a basic AGM contraction, which satisfies* $(\div 1)$–$(\div 6)$

(2) *if any one of* 1.a–1.d *above hold, then* $\bar{\rightsquigarrow}$ *reduces to* $\rightsquigarrow$*:* $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ iff $\alpha \rightsquigarrow \beta$.

Now everything is in place to extend the formalism for a base dependence relation $\bar{\rightsquigarrow}$ that also satisfies CCDF^B. Satisfying CCDF^B allows $\bar{\rightsquigarrow}$ to meet both CCD and CCI.

Theorem 21 (Dependence Generalization with Conjunction). *Let relations* $\bar{\rightsquigarrow}$, $\rightsquigarrow$ *and* $\div$ *for base* B *be such that* Cond$\bar{\rightsquigarrow}$ *and* Cond$\rightsquigarrow$ *hold and* inclusion *is satisfied. In the case where* B *is logically closed,*

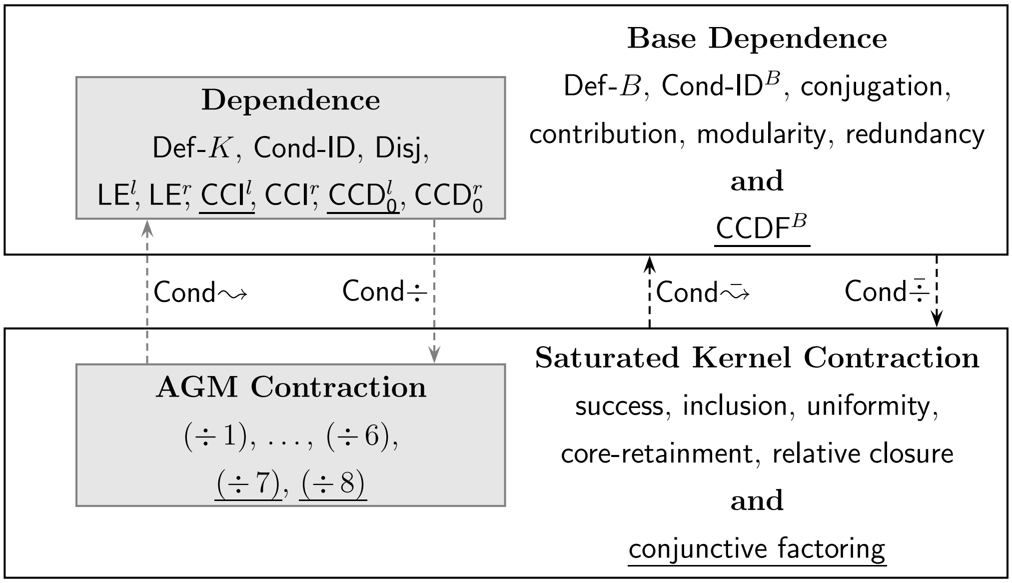

Figure 1: Belief Change and Base Dependence (omitting the underlined axioms results in a weaker characterization)

(1) *the following are logically equivalent:*

a) $\bar{\rightsquigarrow}$ *is a base dependence that satisfies* Def-B, Cond-ID^B, conjugation, contribution, modularity, redundancy *and* CCDF^B

b) $\rightsquigarrow$ *is a dependence, which satisfies* Def-K, Cond-ID, Disj, LE^l, LE^r, CCI^l, CCI^r, CCD$_0^l$ *and* CCD$_0^r$

c) $\div$ *is a saturated kernel contraction that satisfies* success, inclusion, core-retainment, uniformity, relative closure *and* conjunctive factoring

d) $\div$ *is an AGM contraction, which satisfies* $(\div 1)$–$(\div 6)$, $(\div 7)$ *and* $(\div 8)$

(2) *if any one of* 1.a–1.d *above hold, then* $\bar{\rightsquigarrow}$ *reduces to* $\rightsquigarrow$*:* $\alpha \bar{\rightsquigarrow} \beta$ iff $\alpha \rightsquigarrow \beta$.

This concludes our work. The diagram in Figure 1 captures the key results from Theorems 12-21.

Discussion and Related Work

There are several works that define the concepts of relevance and dependence of formulas. Hansson and Wassermann (2002) propose that these can be classified into two groups. Some authors capture relevance/dependence of formulas through syntactical means such as variable sharing and language splitting, including (Parikh 1999; Chopra and Parikh 2000; Makinson and Kourousias 2006; Kourousias and Makinson 2007; Makinson 2007; Ji, Qi, and Haase 2008; Suntisrivaraporn et al. 2008; Ismail and Kasrin 2010; Wu, Zhang, and Zhang 2011; Perrussel, Marchi, and Zhang 2011; Falappa et al. 2011). Other authors have focused on inferential dependency of formulas, or, in other words, how some formulas deductively contribute to inference of other formulas. Examples of this approach include (Farias del Cerro and Herzig 1996; Hansson and Wassermann 2002; Cuenca Grau, Halaschek-Wiener, and Kazakov 2007), as well as the work reported in the present paper. Typically, syntactical approaches are simpler and computationally more efficient compared to inferential approaches. However, the latter usually provide a more accurate and tighter definition of relevance and dependence than syntactical approaches.

What sets FH's work and our work apart from all other works, is the integration with the theory of belief change. This has an important implication: it provides the most theoretically accurate definition of dependence in the context of belief change. For example, because FH construct their dependence relation using AGM contraction, any other definition of dependence that is put forward to be used in relation to AGM contraction is either as good as their dependence relation or less accurate in capturing dependence of formulas in this context. Of course, being a generalization of FH's work, our framework inherits this property for belief sets, and preserves it for belief bases as well.

Conclusion

With these results, we have provided a formalism of Gärdenfors' preservation criterion such that it generalizes the dependence formalism studied by FH so that it works for belief bases (and belief sets). In the case when a belief base is closed the generalized dependence, base dependence, is equivalent to the original FH dependence relation. While generalizing FH's work, it preserves some of the important characteristics of their study such as Keynes' conjunction criterion for dependence (CCD) and Grdenfors' conjunction criterion for independence (CCI). Additionally, we provide a more specific intuition called *conjunction criterion of dependence factoring*, CCDF, that encompasses both Keynes' CCD and Grdenfors' CCI intuitions.

References

Alchourrn, C. E.; Grdenfors, P.; and Makinson, D. 1985. On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic* 50:510530.

Chopra, S., and Parikh, R. 2000. Relevance sensitive belief structures. *Annals of Mathematics and Artificial Intelligence* 28(1):259–285.

Cuenca Grau, B.; Halaschek-Wiener, C.; and Kazakov, Y. 2007. History matters: Incremental ontology reasoning using modules. 183–196.

Falappa, M.; Garca, A.; Kern-Isberner, G.; and Simari, G. 2011. Stratified belief bases revision with argumentative inference. *Journal of Philosophical Logic* 133.

Farias del Cerro, L., and Herzig, A. 1996. Belief change and dependence. In *Proceedings of the 6th conference on Theoretical aspects of rationality and knowledge (TARK'96)*, 147–161. Morgan Kaufmann Publishers Inc. San Francisco, CA, USA.

Grdenfors, P., and Makinson, D. 1988. Revisions of knowledge systems using epistemic entrenchment. In *Proceedings of the 2nd conference on Theoretical aspects of reasoning about knowledge*, 83–95. Pacific Grove, California: Morgan Kaufmann Publishers Inc.

Grdenfors, P. 1978. On the logic of relevance. *Synthese* 37(3):351–367.

Grdenfors, P. 1988. *Knowledge in Flux: Modeling the Dynamics of Epistemic States*. Cambridge, Mass: MIT Press.

Grdenfors, P. 1990. Belief revision and relevance. In *PSA: Proceedings of the Biennial Meeting of the Philosophy of Science Association*, 349–365. Philosophy of Science Association.

Hansson, S. O., and Wassermann, R. 2002. Local change. *Studia Logica* 70(1):4976.

Hansson, S. O. 1993. Theory contraction and base contraction unified. *The Journal of Symbolic Logic* 58(2):602.

Hansson, S. O. 1995. Kernel contraction. *Journal of Symbolic Logic* 845–859.

Hansson, S. O. 1999. *A Textbook of Belief Dynamics: Theory Change and Database Updating*. Kluwer Academic Publishers.

Hansson, S. O. 2003. A dyadic representation of belief. In Grdenfors, P., ed., *Belief Revision*. Cambridge University.

Ismail, H., and Kasrin, N. 2010. High-level perception as focused belief revision. In *Proceeding of the 2010 conference on ECAI 2010: 19th European Conference on Artificial Intelligence*, 11451146.

Ji, Q.; Qi, G.; and Haase, P. 2008. A relevance-based algorithm for finding justifications of DL entailments. Technical report, Technical report, University of Karlsruhe.

Keynes, J. M. 1921. *A Treatise on Probability*. Courier Dover Publications.

Kourousias, G., and Makinson, D. 2007. Parallel interpolation, splitting, and relevance in belief change. *Journal of symbolic logic* 72(3):994–1002.

Makinson, D., and Kourousias, G. 2006. Respecting relevance in belief change. *Anlisis filosfico* 26(1):5361.

Makinson, D. C. 2007. Propositional relevance through letter-sharing: review and contribution. In Bonanno, G.; Delgrande, J.; Lang, J.; and Rott, H., eds., *Formal models of belief change in rational agents*. Dagstuhl, Germany: Internationales Begegnungs und Forschungszentrum fur Informatik (IBFI).

Oveisi, M. 2013. *Belief Change and Base Dependence*. Ph.D. Dissertation, Simon Fraser University, Canada.

Parikh, R. 1999. Beliefs, belief revision, and splitting languages. *Logic, language and computation* 2:266278.

Perrussel, L.; Marchi, J.; and Zhang, D. 2011. Characterizing relevant belief revision operators. *AI 2010: Advances in Artificial Intelligence* 4251.

Suntisrivaraporn, B.; Qi, G.; Ji, Q.; and Haase, P. 2008. A modularization-based approach to finding all justifications for OWL DL entailments. In *Proceedings of the 3rd Asian Semantic Web Conference on The Semantic Web*, 1–15. Springer.

Wu, M.; Zhang, D.; and Zhang, M. 2011. Language splitting and relevance-based belief change in horn logic. In *Twenty-Fifth AAAI Conference on Artificial Intelligence*.