Towards Large-scale Functional Verification of Universal Quantum Circuits, Or: Verifying a Quantum Computing Textbook

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Quantum Physics and Logic
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Outline

Motivation

The path-sum model

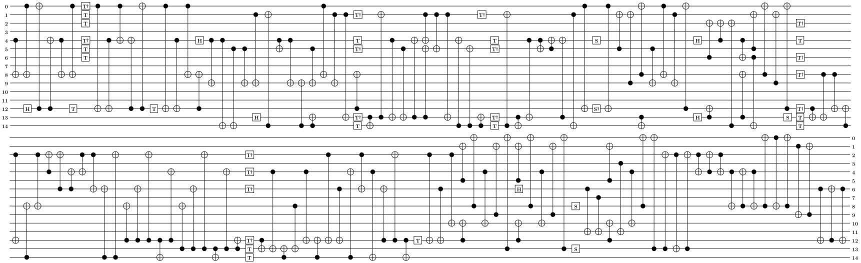
A calculus for path-sums

Completeness

Experimental results

Goal

Automatically verify this:



Against this:

$$|\mathbf{x}\rangle|\mathbf{y}\rangle|\mathbf{0}\rangle \mapsto |\mathbf{x}\rangle|\mathbf{y}\rangle|\mathbf{x}+\mathbf{y}\rangle$$

Motivation, Pt I

Specification

How should the functionality be specified?

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- Matrix?

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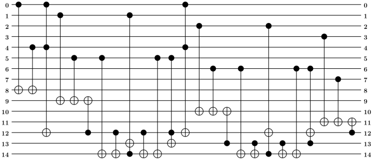
- Higher-level circuit?



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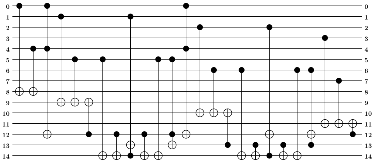

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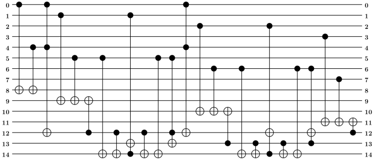

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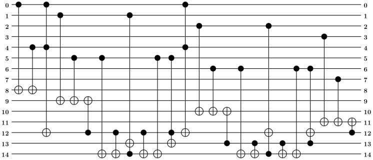

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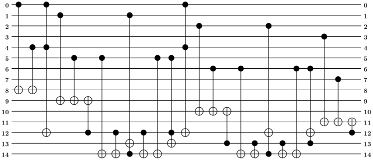

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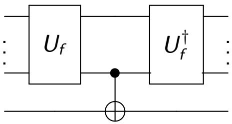
Bottom line: $|\mathbf{x}\rangle|\mathbf{x}\rangle|\mathbf{0}\rangle \mapsto |\mathbf{x}\rangle|\mathbf{y}\rangle|\mathbf{x}+\mathbf{y}\rangle$ concisely captures the intuition

Motivation, Pt II

Target circuits

Theory:



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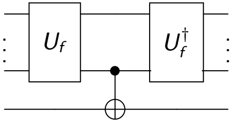

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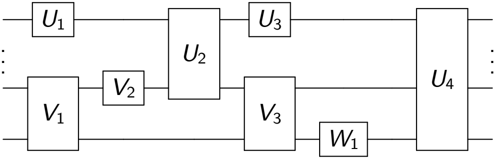

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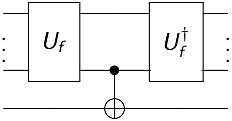

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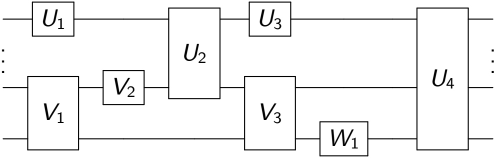

Optimizations are really hard to formally prove correct

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- Natural to write specifications for quantum algorithms
- Poly-time computable for fixed levels of the Clifford hierarchy
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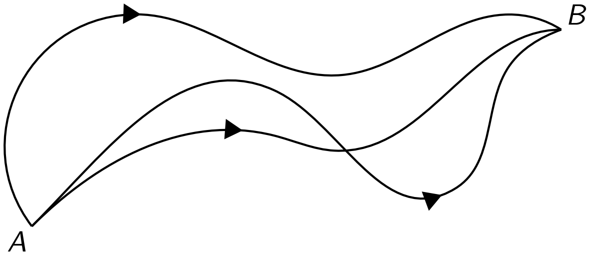
- Admits a natural notion of reduction
- Only computational paths matter!

The Feynman path integral

Amplitude of a quantum state is a sum over all paths leading to it



Path-sums

phase polynomials on steroids

$$H : \quad |x\rangle \quad \mapsto \frac{1}{\sqrt{2}} \textstyle\sum_{y \in \mathbb{Z}_2} \quad e^{2\pi i \frac{xy}{2}} \quad |y\rangle$$

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[0] [Dawson et. al., 2004]

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Note: *well-formed* $=$ *partial isometry*

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Examples

$$T : |x\rangle \mapsto e^{2\pi i \frac{x}{8}} |x\rangle$$

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$$\text{Toffoli}_n : |x_1 x_2 \cdots x_n\rangle \mapsto |x_1 x_2 \cdots (x_n \oplus \prod_{i=1}^{n-1} x_i)\rangle$$

$$\text{Adder}_n : |\mathbf{x}\rangle |\mathbf{y}\rangle |\mathbf{0}\rangle \mapsto |\mathbf{x}\rangle |\mathbf{y}\rangle |\mathbf{x} + \mathbf{y}\rangle$$

$$\text{QFT}_n : |\mathbf{x}\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{\mathbf{y} \in \mathbb{Z}_2^n} e^{2\pi i \frac{[\mathbf{x} \cdot \mathbf{y}]}{2^n}} |\mathbf{y}\rangle$$

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Composing path sums

$$\xi = |\mathbf{x}\rangle \mapsto \frac{1}{\sqrt{2^m}} \sum_{\mathbf{y}\in\mathbb{Z}_2^m} e^{2\pi i P(\mathbf{x},\mathbf{y})} |f(\mathbf{x},\mathbf{y})\rangle$$

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Tensor:

$$\xi \otimes \xi' = |\mathbf{x}\rangle|\mathbf{x}'\rangle \mapsto \frac{1}{\sqrt{2^{m+m'}}} \sum_{\mathbf{y}\in\mathbb{Z}_2^m,\mathbf{y}'\in\mathbb{Z}_2^{m'}} e^{2\pi i \left(P(\mathbf{x}\,\mathbf{y})+P'(\mathbf{x}',\mathbf{y}')\right)} |f(\mathbf{x},\mathbf{y})\rangle|f'(\mathbf{x}',\mathbf{y}')\rangle$$

Functional:

$$\xi' \circ \xi = ???$$

Functional composition

$$|x_1'x_2'x_3'\rangle \mapsto |x_1'x_2'(x_2' \oplus x_3')\rangle \circ |x_1x_2x_3\rangle \mapsto |x_1(x_1 \oplus x_2)x_3\rangle$$

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Functional composition

Composing isometries

What about the following composition?

$$|0\rangle \mapsto |0\rangle \circ |x\rangle \mapsto \frac{1}{\sqrt{2}} \sum_{y \in \mathbb{Z}_2} e^{\pi i x y} |y\rangle$$

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An output signature is $|f(\mathbf{x}, \mathbf{y})\rangle$ compatible with an input signature $|\mathbf{x}'\rangle$ if and only if whenever $x'_i = 0$ or 1, $f_i(\mathbf{x}, \mathbf{y}) = x'_i$

> *E.g.* $|1\rangle$ *is compatible with* $|x\rangle$ *while* $|x\rangle$ *is not compatible with* $|1\rangle$

Functional composition

Substitutions inside phase polynomials

$$|x\rangle \mapsto e^{2\pi i \frac{x}{4}}|x\rangle \circ |x\rangle \mapsto |1 \oplus x\rangle$$

Need to lift the Boolean polynomial $1 \oplus x$ to a functionally equivalent polynomial $\overline{1 \oplus x}$ over dyadic fractions

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$$\begin{aligned}\overline{\mathbf{x}^{\alpha}} &= \mathbf{x}^{\alpha}, \\ \overline{P + Q} &= \overline{P} + \overline{Q} - 2\overline{PQ},\end{aligned}$$

Proposition

For any Boolean-valued polynomial P and all $\mathbf{x} \in \mathbb{Z}_2^n$, $\overline{P}(\mathbf{x}) = P(\mathbf{x})$ mod 2*.*

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The path-sum model

Path-sum semantics for Clifford+R_k circuits:

$$\begin{aligned}
[\![H]\!] &= |x\rangle \mapsto \frac{1}{\sqrt{2}} \sum_{y\in\{0,1\}} e^{2\pi i \frac{xy}{2}} |y\rangle \\
[\![R_k]\!] &= |x\rangle \mapsto e^{2\pi i \frac{x}{2^k}} |x\rangle \\
[\![R_k^\dagger]\!] &= |x\rangle \mapsto e^{2\pi i \frac{-x}{2^k}} |x\rangle \\
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\end{aligned}
$$

Proposition

The path-sum of an n-qubit Clifford+R_k circuit C for fixed k has size polynomial in the volume of C and can be computed in polynomial time.

Digression: only computational paths matter

The path-sum model normalizes[1] most structural equivalences, as well as some semantic equivalences.

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[1] Caveat: up to variable renaming

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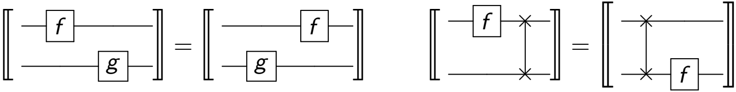

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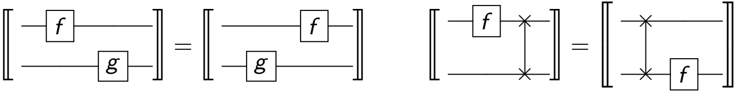
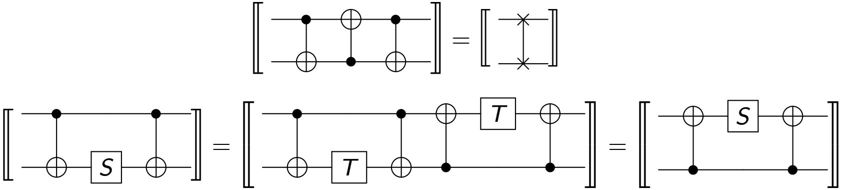


Semantic equivalences:



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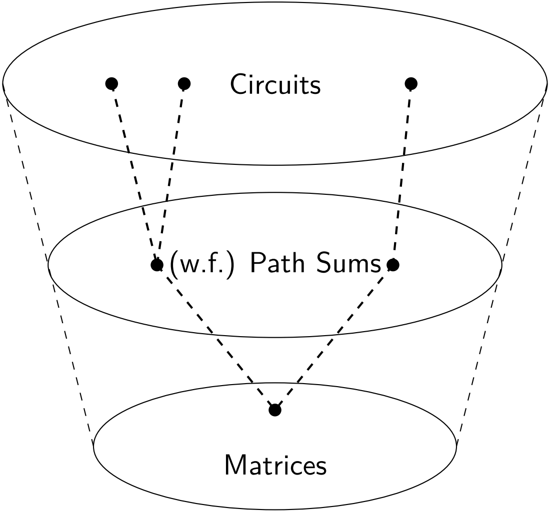
Path-sums as an intermediary model



Motivation

The path-sum model

A calculus for path-sums

Completeness

Experimental results

Reducing path-sums

- Path-sums are an un-evaluated representation of the branching computational paths in a circuit
- Lose any computational advantage if we just expand all paths
- Instead, find groups of paths which interfere in recognizable ways

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reduction $\equiv$ *path variable elimination*

Example

$HH = I$

$$HH : |x\rangle \mapsto \frac{1}{2} \sum_{y_1, y_2 \in \mathbb{Z}_2} e^{2\pi i \frac{xy_1 + y_1 y_2}{2}} |y_2\rangle$$

Example

$HH = I$

$$\frac{1}{2} \sum_{y_1, y_2 \in \mathbb{Z}_2} e^{2\pi i \frac{x y_1 + y_1 y_2}{2}} \left| y_2 \right\rangle$$

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$$\begin{tikzcd}
 & \frac{1}{2}\sum_{y_1,y_2\in\mathbb{Z}_2} e^{2\pi i\frac{xy_1+y_1y_2}{2}}\left|y_2\right\rangle \arrow[dl, "y_1=0"'] \arrow[dr, "y_1=1"] & \\
\frac{1}{2}\sum_{y_2\in\mathbb{Z}_2}\left|y_2\right\rangle & & \frac{1}{2}\sum_{y_2\in\mathbb{Z}_2} e^{2\pi i\frac{x+y_2}{2}}\left|y_2\right\rangle
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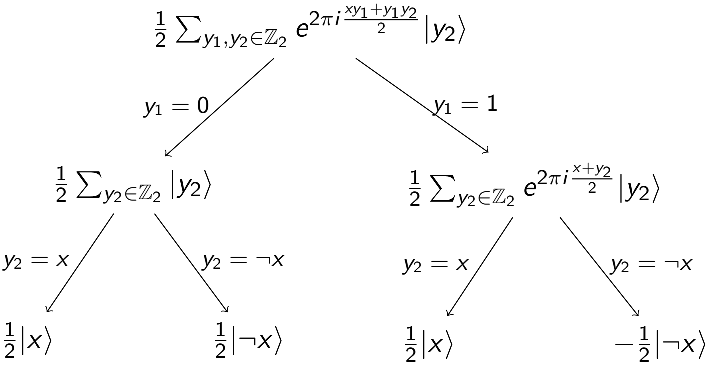

$$\frac{1}{2}\sum_{y_1,y_2\in\mathbb{Z}_2} e^{2\pi i\frac{xy_1+y_1y_2}{2}}|y_2\rangle$$

$y_1 = 0$

$y_1 = 1$

$$\frac{1}{2}\sum_{y_2\in\mathbb{Z}_2}|y_2\rangle$$

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$y_2 = x$

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$\frac{1}{2}|x\rangle$

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 & \frac{1}{2}\sum_{y_2\in\mathbb{Z}_2}\left|y_2\right\rangle \arrow[dl, "y_2 = x"'] \arrow[dr, "y_2 = \neg x"] & & & & \frac{1}{2}\sum_{y_2\in\mathbb{Z}_2} e^{2\pi i\frac{x+y_2}{2}}\left|y_2\right\rangle \arrow[dl, "y_2 = x"'] \arrow[dr, "y_2 = \neg x"] & \\
\frac{1}{2}\left|x\right\rangle \arrow[drr] & & \frac{1}{2}\left|\neg x\right\rangle \arrow[drr] & & \frac{1}{2}\left|x\right\rangle \arrow[dll] & & -\frac{1}{2}\left|\neg x\right\rangle \arrow[dll] \\
 & & \frac{2}{2}\left|x\right\rangle & + & \frac{0}{2}\left|\neg x\right\rangle & &
\end{tikzcd}$$

Generalization

Whenever

$$P(\mathbf{x}, \mathbf{y}) = \frac{1}{2} y_0 (y_i + Q(\mathbf{x}, \mathbf{y})) + R(\mathbf{x}, \mathbf{y})$$

for an internal path variable y_0, $y_i \notin Q$ and Q Boolean,

- the paths defined by $y_i = Q(\mathbf{x}, \mathbf{y})$, $y_0 = 0$ and $y_0 = 1$ add, and
- the paths defined by $y_i = \neg Q(\mathbf{x}, \mathbf{y})$, $y_0 = 0$ and $y_0 = 1$ cancel

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Equationally,

$$\begin{aligned} &\tfrac{1}{\sqrt{2^{m+1}}} \textstyle\sum_{y_0\in\mathbb{Z}_2} \sum_{\mathbf{y}\in\mathbb{Z}_2^m} e^{2\pi i\left(\frac{1}{2}y_0(y_i+Q(\mathbf{x},\mathbf{y}))+R(\mathbf{x},\mathbf{y})\right)}|f(\mathbf{x},\mathbf{y})\rangle \\ &= \tfrac{1}{\sqrt{2^{m+1}}} \textstyle\sum_{\mathbf{y}\in\mathbb{Z}_2^m} e^{2\pi i\left(R[y_i\leftarrow\overline{Q}]\right)(\mathbf{x},\mathbf{y})}|\left(f[y_i \leftarrow Q]\right)(\mathbf{x},\mathbf{y})\rangle \end{aligned}$$

Rewrite rules

$$\frac{1}{\sqrt{2^{m+2}}}\sum_{y_0\in\mathbb{Z}_2}\sum_{\mathbf{y}\in\mathbb{Z}_2^m} e^{2\pi i P(\mathbf{x},\mathbf{y})}|f(\mathbf{x},\mathbf{y})\rangle \longrightarrow \frac{1}{\sqrt{2^m}}\sum_{\mathbf{y}\in\mathbb{Z}_2^m} e^{2\pi i P(\mathbf{x},\mathbf{y})}|f(\mathbf{x},\mathbf{y})\rangle \qquad \text{[Elim]}$$

$$\frac{1}{\sqrt{2^{m+1}}}\sum_{y_0\in\mathbb{Z}_2}\sum_{\mathbf{y}\in\mathbb{Z}_2^m} e^{2\pi i\left(\frac{1}{4}y_0+\frac{1}{2}y_0 Q(\mathbf{x},\mathbf{y})+R(\mathbf{x},\mathbf{y})\right)}|f(\mathbf{x},\mathbf{y})\rangle \longrightarrow \frac{1}{\sqrt{2^m}}\sum_{y\in\mathbb{Z}_2^m} e^{2\pi i\left(\frac{1}{8}-\frac{1}{4}\overline{Q}(\mathbf{x},\mathbf{y})+R(\mathbf{x},\mathbf{y})\right)}|f(\mathbf{x},\mathbf{y})\rangle \qquad [\omega]$$

$$\frac{1}{\sqrt{2^{m+1}}}\sum_{y_0\in\mathbb{Z}_2}\sum_{\mathbf{y}\in\mathbb{Z}_2^m} e^{2\pi i\left(\frac{1}{2}y_0(y_i+Q(\mathbf{x},\mathbf{y}))+R(\mathbf{x},\mathbf{y})\right)}|f(\mathbf{x},\mathbf{y})\rangle \longrightarrow \frac{1}{\sqrt{2^{m+1}}}\sum_{\mathbf{y}\in\mathbb{Z}_2^m} e^{2\pi i\left(R[y_i\leftarrow\overline{Q}]\right)(\mathbf{x},\mathbf{y})}|\,(f[y_i\leftarrow Q])\,(\mathbf{x},\mathbf{y})\rangle \qquad \text{[HH]}$$

$$\frac{P(\mathbf{x},\mathbf{y}) = \frac{1}{4}y_i x + \frac{1}{2}y_i(y_j + Q(\mathbf{x},\mathbf{y})) + R(\mathbf{x},\mathbf{y}) = \frac{1}{4}y_j(1-x) + \frac{1}{2}y_j(y_i + Q'(\mathbf{x},\mathbf{y})) + R'(\mathbf{x},\mathbf{y})}{\frac{1}{\sqrt{2^{m+2}}}\sum_{\mathbf{y}\in\mathbb{Z}_2^{m+2}} e^{2\pi i P(\mathbf{x},\mathbf{y})}|f(\mathbf{x},\mathbf{y})\rangle \longrightarrow \frac{1}{\sqrt{2^m}}\sum_{\mathbf{y}\in\mathbb{Z}_2^m} e^{2\pi i\left((1-x)R[y_j\leftarrow\overline{Q}]+xR'[y_i\leftarrow\overline{Q'}]\right)(\mathbf{x},\mathbf{y})}|f(\mathbf{x},\mathbf{y})\rangle} \qquad \text{[Case]}$$

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$$\frac{1}{\sqrt{2^{m+2}}}\sum_{y_0\in\mathbb{Z}_2}\sum_{\mathbf{y}\in\mathbb{Z}_2^m} e^{2\pi i P(\mathbf{x},\mathbf{y})}|f(\mathbf{x},\mathbf{y})\rangle \longrightarrow \frac{1}{\sqrt{2^m}}\sum_{\mathbf{y}\in\mathbb{Z}_2^m} e^{2\pi i P(\mathbf{x},\mathbf{y})}|f(\mathbf{x},\mathbf{y})\rangle \qquad \text{[Elim]}$$

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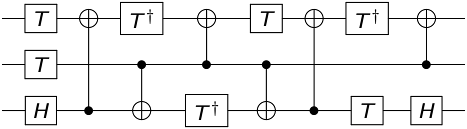
Key property: number of path variables are always reduced!

Example

Toffoli$_3$: $|x_1 x_2 x_3\rangle \mapsto |x_1 x_2 (x_3 \oplus x_1 x_2)\rangle$



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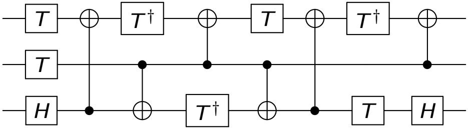

$$|x_1x_2x_3\rangle \mapsto \frac{1}{\sqrt{2^2}} \sum_{y_1,y_2\in\mathbb{Z}_2} e^{2\pi i \frac{1}{2}(x_3 y_1 + x_1x_2 y_1 + y_1 y_2)} |x_1x_2y_2\rangle$$

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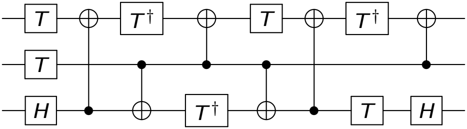

$$|x_1x_2x_3\rangle \mapsto \frac{1}{\sqrt{2^2}} \sum_{y_1,y_2\in\mathbb{Z}_2} e^{2\pi i \frac{1}{2}(x_3 y_1 + x_1x_2 y_1 + y_1 y_2)} |x_1x_2y_2\rangle$$

$$\mapsto \frac{1}{\sqrt{2^2}} \sum_{y_1,y_2\in\mathbb{Z}_2} e^{2\pi i \frac{1}{2} y_1 (y_2 + x_3 + x_1x_2)} |x_1x_2y_2\rangle$$

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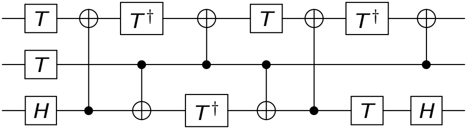

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\begin{aligned}
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&\mapsto \frac{1}{\sqrt{2^2}} \sum_{y_1,y_2\in\mathbb{Z}_2} e^{2\pi i \frac{1}{2} y_1 (y_2 + x_3 + x_1x_2)} |x_1x_2y_2\rangle \\
&\mapsto \frac{1}{\sqrt{2^2}} \sum_{y_2\in\mathbb{Z}_2} |x_1x_2(x_3 \oplus x_1x_2)\rangle \qquad [\mathrm{HH},\ y_2 \leftarrow x_3 \oplus x_1x_2]
\end{aligned}
$$

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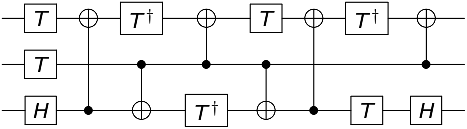

$$\begin{aligned}
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&\mapsto \frac{1}{\sqrt{2^2}} \sum_{y_1,y_2\in\mathbb{Z}_2} e^{2\pi i \frac{1}{2}y_1(y_2+x_3+x_1x_2)}|x_1x_2y_2\rangle \\
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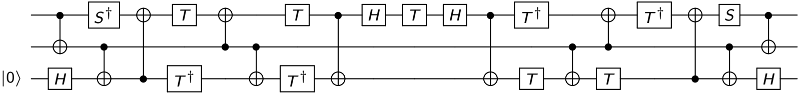
&\mapsto |x_1x_2(x_3 \oplus x_1x_2)\rangle && [\text{Elim } y_2]
\end{aligned}$$

Example

Controlled-T : $|x_1 x_2\rangle \mapsto e^{2\pi i \frac{x_1 x_2}{8}} |x_1 x_2\rangle$



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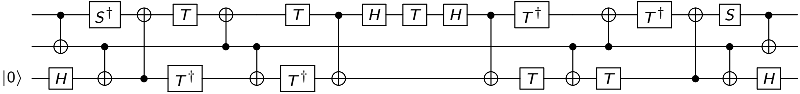

$$|x_1 x_2\rangle|0\rangle \mapsto \frac{1}{\sqrt{2^4}} \sum_{\mathbf{y}\in\mathbb{Z}_2^4} e^{2\pi i \frac{1}{8}(4x_1x_2y_1+4x_1y_2+4y_1y_2+y_2+4y_2y_3+4x_1x_2y_3+4x_1y_4+4y_3y_4+4x_1x_2)}|x_1 x_2 y_4\rangle$$

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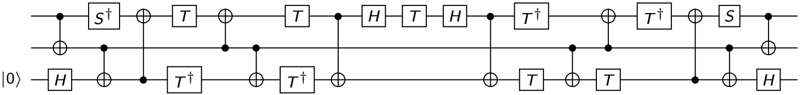

$$
\begin{aligned}
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&\mapsto \frac{1}{\sqrt{2^4}} \sum_{\mathbf{y}\in\mathbb{Z}_2^4} e^{2\pi i \left(\frac{1}{2}y_1(y_2+x_1x_2)+\frac{1}{8}(4x_1y_2+y_2+4y_2y_3+4x_1x_2y_3+4x_1y_4+4y_3y_4+4x_1x_2)\right)} |x_1x_2y_4\rangle
\end{aligned}
$$

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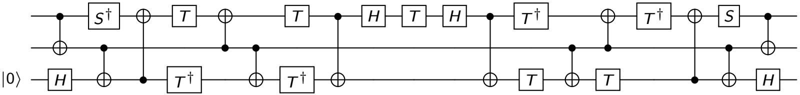
Controlled-T : $|x_1x_2\rangle \mapsto e^{2\pi i \frac{x_1x_2}{8}} |x_1x_2\rangle$



$$|x_1x_2\rangle|0\rangle \mapsto \frac{1}{\sqrt{2^4}} \sum_{\mathbf{y}\in\mathbb{Z}_2^4} e^{2\pi i \frac{1}{8}(4x_1x_2y_1+4x_1y_2+4y_1y_2+y_2+4y_2y_3+4x_1x_2y_3+4x_1y_4+4y_3y_4+4x_1x_2)}|x_1x_2y_4\rangle$$

$$\mapsto \frac{1}{\sqrt{2^4}} \sum_{\mathbf{y}\in\mathbb{Z}_2^4} e^{2\pi i\left(\frac{1}{2}y_1(y_2+x_1x_2)+\frac{1}{8}(4x_1y_2+y_2+4y_2y_3+4x_1x_2y_3+4x_1y_4+4y_3y_4+4x_1x_2)\right)}|x_1x_2y_4\rangle$$

$$\mapsto \frac{1}{\sqrt{2^2}} \sum_{y_3,y_4\in\mathbb{Z}_2} e^{2\pi i \frac{1}{8}(4x_1x_2+x_1x_2+4x_1x_2y_3+4x_1x_2y_3+4x_1y_4+4y_3y_4+4x_1x_2)}|x_1x_2y_4\rangle \quad \text{[HH, Elim]}$$

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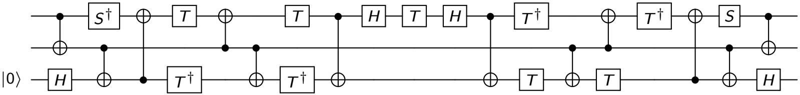


$$|x_1x_2\rangle|0\rangle \mapsto \frac{1}{\sqrt{2^4}}\sum_{\mathbf{y}\in\mathbb{Z}_2^4} e^{2\pi i \frac{1}{8}(4x_1x_2{\color{red}y_1}+4x_1y_2+4{\color{red}y_1}y_2+y_2+4y_2y_3+4x_1x_2y_3+4x_1y_4+4y_3y_4+4x_1x_2)}|x_1x_2y_4\rangle$$

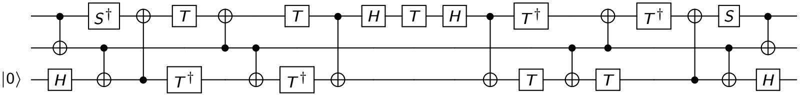
$$\mapsto \frac{1}{\sqrt{2^4}}\sum_{\mathbf{y}\in\mathbb{Z}_2^4} e^{2\pi i\left(\frac{1}{2}{\color{red}y_1}(y_2+x_1x_2)+\frac{1}{8}(4x_1y_2+y_2+4y_2y_3+4x_1x_2y_3+4x_1y_4+4y_3y_4+4x_1x_2)\right)}|x_1x_2y_4\rangle$$

$$\mapsto \frac{1}{\sqrt{2^2}}\sum_{y_3,y_4\in\mathbb{Z}_2} e^{2\pi i \frac{1}{8}(4x_1x_2+x_1x_2+4x_1x_2{\color{red}y_3}+4x_1x_2{\color{red}y_3}+4x_1y_4+4{\color{red}y_3}y_4+4x_1x_2)}|x_1x_2y_4\rangle \quad \text{[HH, Elim]}$$

$$\mapsto \frac{1}{\sqrt{2^2}}\sum_{y_3,y_4\in\mathbb{Z}_2} e^{2\pi i\left(\frac{1}{2}{\color{red}y_3}y_4+\frac{1}{8}(x_1y_4+x_1x_2)\right)}|x_1x_2y_4\rangle$$

Example

Controlled-T : $|x_1 x_2\rangle \mapsto e^{2\pi i \frac{x_1 x_2}{8}} |x_1 x_2\rangle$



$$|x_1 x_2\rangle|0\rangle \mapsto \frac{1}{\sqrt{2^4}} \sum_{\mathbf{y}\in\mathbb{Z}_2^4} e^{2\pi i \frac{1}{8}(4x_1x_2y_1 + 4x_1y_2 + 4y_1y_2 + y_2 + 4y_2y_3 + 4x_1x_2y_3 + 4x_1y_4 + 4y_3y_4 + 4x_1x_2)} |x_1 x_2 y_4\rangle$$

$$\mapsto \frac{1}{\sqrt{2^4}} \sum_{\mathbf{y}\in\mathbb{Z}_2^4} e^{2\pi i \left(\frac{1}{2}y_1(y_2 + x_1x_2) + \frac{1}{8}(4x_1y_2 + y_2 + 4y_2y_3 + 4x_1x_2y_3 + 4x_1y_4 + 4y_3y_4 + 4x_1x_2)\right)} |x_1 x_2 y_4\rangle$$

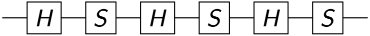
$$\mapsto \frac{1}{\sqrt{2^2}} \sum_{y_3, y_4 \in\mathbb{Z}_2} e^{2\pi i \frac{1}{8}(4x_1x_2 + x_1x_2 + 4x_1x_2y_3 + 4x_1x_2y_3 + 4x_1y_4 + 4y_3y_4 + 4x_1x_2)} |x_1 x_2 y_4\rangle \quad \text{[HH, Elim]}$$

$$\mapsto \frac{1}{\sqrt{2^2}} \sum_{y_3, y_4 \in\mathbb{Z}_2} e^{2\pi i \left(\frac{1}{2}y_3y_4 + \frac{1}{8}(x_1y_4 + x_1x_2)\right)} |x_1 x_2 y_4\rangle$$

$$\mapsto e^{2\pi i \frac{x_1 x_2}{8}} |x_1 x_2\rangle|0\rangle \qquad \text{[HH, Elim]}$$

Example

$(\mathrm{SH})^3 : |x\rangle \mapsto \omega|x\rangle$



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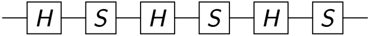

$$(\mathrm{SH})^3 : |x\rangle \mapsto \frac{1}{\sqrt{2}^3} \sum_{y_1,y_2,y_3\in\mathbb{Z}_2} e^{2\pi i \frac{1}{8}(4xy_1+2y_1+4y_1y_2+2y_2+4y_2y_3+2y_3)}|y_3\rangle$$

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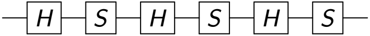
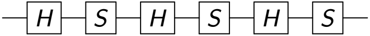


$$
\begin{aligned}
(\mathrm{SH})^3 : |x\rangle &\mapsto \frac{1}{\sqrt{2}^3} \sum_{y_1, y_2, y_3 \in \mathbb{Z}_2} e^{2\pi i \frac{1}{8}(4x y_1 + 2y_1 + 4y_1 y_2 + 2y_2 + 4y_2 y_3 + 2y_3)} |y_3\rangle \\
&\mapsto \frac{1}{\sqrt{2}^3} \sum_{y_1, y_2, y_3 \in \mathbb{Z}_2} e^{2\pi i \left(\frac{1}{4} y_1 + \frac{1}{2} y_1 (y_2 + x) + \frac{1}{8}(2y_2 + 4y_2 y_3 + 2y_3) \right)} |y_3\rangle
\end{aligned}
$$

Example

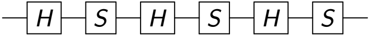
$(\mathrm{SH})^3 : |x\rangle \mapsto \omega|x\rangle$



$$\begin{aligned}
(\mathrm{SH})^3 : |x\rangle &\mapsto \frac{1}{\sqrt{2}^3} \sum_{y_1,y_2,y_3\in\mathbb{Z}_2} e^{2\pi i \frac{1}{8}(4xy_1+2y_1+4y_1y_2+2y_2+4y_2y_3+2y_3)}|y_3\rangle \\
&\mapsto \frac{1}{\sqrt{2}^3} \sum_{y_1,y_2,y_3\in\mathbb{Z}_2} e^{2\pi i\left(\frac{1}{4}y_1+\frac{1}{2}y_1(y_2+x)+\frac{1}{8}(2y_2+4y_2y_3+2y_3)\right)}|y_3\rangle \\
&\mapsto \frac{1}{\sqrt{2}^2} \sum_{y_2,y_3\in\mathbb{Z}_2} e^{2\pi i \frac{1}{8}(1-2(y_2+x-2y_2x)+2y_2+4y_2y_3+2y_3)}|y_3\rangle \qquad [\omega]
\end{aligned}$$

Example

$(\mathrm{SH})^3 : |x\rangle \mapsto \omega|x\rangle$



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(\mathrm{SH})^3 : |x\rangle &\mapsto \frac{1}{\sqrt{2}^3} \sum_{y_1, y_2, y_3 \in \mathbb{Z}_2} e^{2\pi i \frac{1}{8}(4xy_1 + 2y_1 + 4y_1y_2 + 2y_2 + 4y_2y_3 + 2y_3)} |y_3\rangle \\
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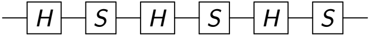

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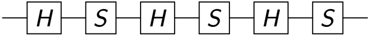

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Motivation

The path-sum model

A calculus for path-sums

Completeness

Experimental results

Completeness

Linear number of steps to reach an irreducible form

$\Longrightarrow$ *incomplete in general, in the sense that normal forms are not unique*

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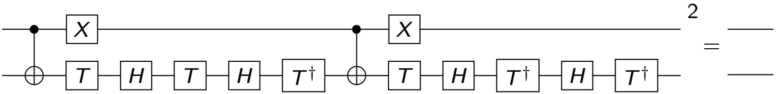
E.g. [Selinger and Bian, 2016]



not provable with current set of rules

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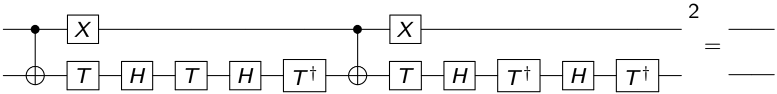


not provable with current set of rules

However, complete for Clifford group with a little extra work

Output restriction

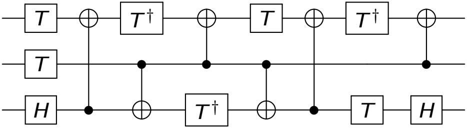
Observation:

If ξ is an isometry then $\xi \equiv |\mathbf{x}\rangle \mapsto |\mathbf{x}'\rangle$ if and only if

$$\frac{1}{\sqrt{2^m}} \sum_{\mathbf{y}\ s.t.\ f(\mathbf{x},\mathbf{y})=\mathbf{x}'} e^{2\pi i P(\mathbf{x},\mathbf{y})} = 1$$

Example

Toffoli redux



$$|x_1 x_2 x_3\rangle \mapsto \frac{1}{2}\sum_{y_1,y_2\in\mathbb{Z}_2} e^{2\pi i \frac{1}{2}(x_3 y_1 + x_1 x_2 y_1 + y_1 y_2)} |x_1 x_2 y_2\rangle$$

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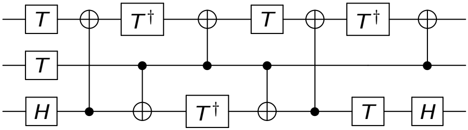

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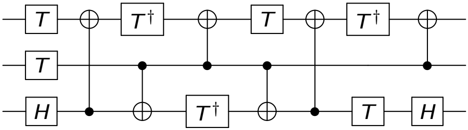

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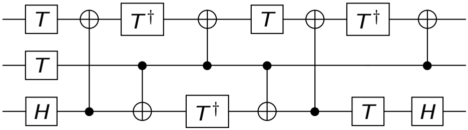

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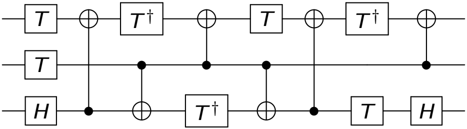

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\end{aligned}$$

Non-equivalence

Observation:

In the LHS of [HH],

$$\frac{1}{\sqrt{2^{m+1}}}\sum_{y_0\in\mathbb{Z}_2}\sum_{\mathbf{y}\in\mathbb{Z}_2^m} e^{2\pi i\left(\frac{1}{2}y_0 Q(\mathbf{x},\mathbf{y})+R(\mathbf{x},\mathbf{y})\right)}|f(\mathbf{x},\mathbf{y})\rangle$$

if Q contains only input variables, then there exists an input basis state $\mathbf{x}$ *such that* $Q(\mathbf{x},\mathbf{y}) = 1 \mod 2$ *for all* $\mathbf{y}$*, so*

$$\frac{1}{\sqrt{2^{m+1}}}\sum_{y_0\in\mathbb{Z}_2}\sum_{\mathbf{y}\in\mathbb{Z}_2^m} e^{2\pi i\left(\frac{1}{2}y_0 Q(\mathbf{x},\mathbf{y})+R(\mathbf{x},\mathbf{y})\right)}|f(\mathbf{x},\mathbf{y})\rangle = 0$$

(Semi)-Completeness for Clifford group circuits

Theorem

Equivalence of Clifford group circuits can be checked in polynomial time.

Proof sketch

Proof sketch

1. Reduce to checking identity of Clifford path sum with form

$$|\mathbf{x}\rangle \mapsto \frac{1}{\sqrt{2^m}} \sum_{\mathbf{y}\in\mathbb{Z}_2^m} e^{2\pi i P(\mathbf{x},\mathbf{y})} |\mathbf{x}\rangle$$

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   - I.e. output restriction observation

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    - Clifford path-sum has phase polynomial of degree ≤ 2
    - Reductions don't increase degree of P when $\deg(P) \leq 2$
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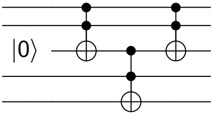
Implementation

https://github.com/meamy/feynman

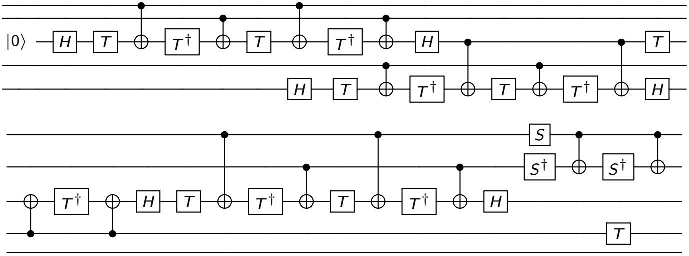
- Written in Haskell
- $\sim$ 500 lines of code
- No real language for specifying path-sums currently

Translation validation

Original (Tof_3):



Optimized:

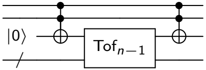


Suite of 38 benchmarks averaging 24 qubits

- 31 passing, 4 failing, 3 did not finish
- Largest completed: 96 qubits, 252 path variables, 25k gates in 530s
- Runs out of memory ∼1000 variables

Functional verification

Toff$_n$: $|x_1 x_2 \cdots x_n\rangle \mapsto |x_1 x_2 \cdots (x_n \oplus \prod_{i=1}^{n-1} x_i)\rangle$



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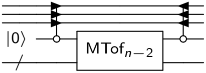
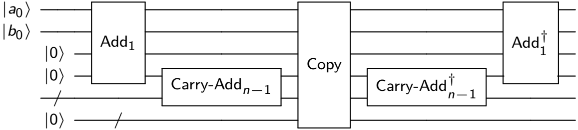
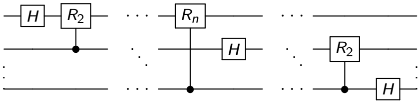


[Maslov 2016]

Adder$_n$: $|\mathbf{x}\rangle|\mathbf{y}\rangle|\mathbf{0}\rangle \mapsto |\mathbf{x}\rangle|\mathbf{y}\rangle|\mathbf{x}+\mathbf{y}\rangle$

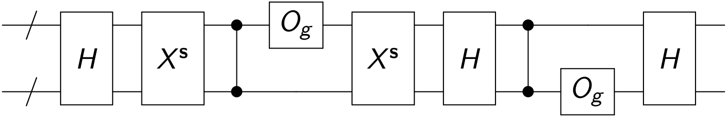


QFT$_n$: $|\mathbf{x}\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{\mathbf{y}\in\mathbb{Z}_2^n} e^{2\pi i \frac{[\mathbf{x}\cdot\mathbf{y}]}{2^n}} |\mathbf{y}\rangle$



Hidden shift

Quantum algorithm to find a hidden shift vector $\mathbf{s}$ *for a pair of shifted Maiorana-McFarland bent functions [Roetteler 2010]*



- Implements transformation $|\mathbf{0}\rangle \mapsto |\mathbf{s}\rangle$
- O_g randomly generated with A CCZ gates and $200 \cdot A$ $\{Z, CZ\}$ gates

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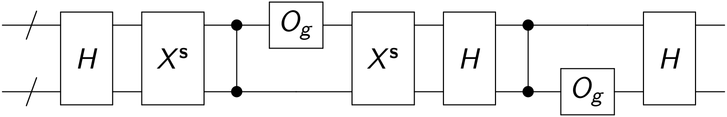

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Simulation $(n = 40, A = 5)$ in 4s, vs. hours [Brayvi & Gosset 2016]

Results

Algorithm	n	m	Clifford	T	Result	Time (s)
Toffoli_{50}	97	190	855	665	PASS	1.078
Toffoli_{100}	197	390	1755	1365	PASS	5.346
Maslov_{50}	74	192	481	384	PASS	0.759
Maslov_{100}	149	392	981	784	PASS	3.937
Adder_{8}	40	56	334	196	PASS	0.142
Adder_{16}	80	120	710	420	PASS	26.151
QFT_{16}	16	16	256	–	PASS	1.250
QFT_{31}	31	31	961	–	PASS	16.929
$\text{Hidden Shift}_{20,4}$	20	60	5254	56	PASS	1.064
$\text{Hidden Shift}_{40,5}$	40	120	6466	70	PASS	3.573
$\text{Hidden Shift}_{60,10}$	60	180	12784	140	PASS	12.811
$\text{Symbolic Shift}_{20,4}$	40	60	5296	56	PASS	1.877
$\text{Symbolic Shift}_{40,5}$	80	120	6638	70	PASS	6.633
$\text{Symbolic Shift}_{60,10}$	120	180	12804	140	PASS	34.840

Conclusion

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- A calculus for reducing path-sums
- A verification method which is complete for Clifford group circuits

Future work

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- Investigate use as a proof technique in inductive & higher order proofs

Thank you!