Symbolic representation of quantum computations

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Algebraic Structures in Quantum Computation
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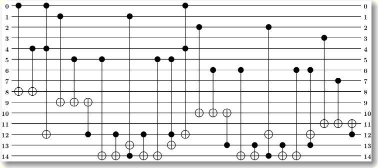
What is this talk about?

How can computers (and humans) express, reason *about, and* understand *quantum computations?*

Representations of quantum computations

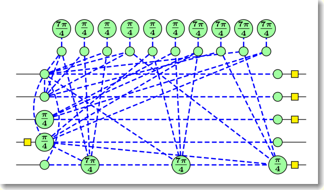
Standard representations are based around **composing linear operators**

- Circuits - composing unitary matrices (+ measurement)
- Quantum programming languages - complicated classical code describing a circuit
- ZX-calculus - tensor networks over a particular basis




```
%Array = type opaque
%Result = type opaque
%Qubit = type opaque

declare %Result* @__quantum__qis__mz(%Qubit* %0)
declare void @__quantum__qis__cnot(%Qubit* %0, %Qubit* %1)
declare void @__quantum__qis__h(%Qubit* %0)

declare %Array* @__quantum__rt__qubit_allocate_array(i64 %0)
declare void @__quantum__rt__qubit_release_array(%Array* %0)
declare i8* @__quantum__rt__array_get_element_ptr_1d(%Array* %0, i64 %1)

define i32 @main(i32 %0, i8** %1)  {
  %4 = call %Array* @__quantum__rt__qubit_allocate_array(i64 2)
  %5 = call i8* @__quantum__rt__array_get_element_ptr_1d(%Array* %4, i64 0)
  %6 = bitcast i8* %5 to %Qubit**
  %7 = load %Qubit*, %Qubit** %6, align 8
  call void @__quantum__qis__h(%Qubit* %7)
  %8 = call i8* @__quantum__rt__array_get_element_ptr_1d(%Array* %4, i64 1)
  %9 = bitcast i8* %8 to %Qubit**
  %10 = load %Qubit*, %Qubit** %9, align 8
  call void @__quantum__qis__cnot(%Qubit* %7, %Qubit* %10)
  %11 = call %Result* @__quantum__qis__mz(%Qubit* %7)
  %12 = call %Result* @__quantum__qis__mz(%Qubit* %10)
  call void @__quantum__rt__qubit_release_array(%Array* %4)
  ret i32 0
}
```

```
OPENQASM 2.0;
include "qelib1.inc";

qreg q[5];

ccx q[0],q[1],q[4];
ccx q[2],q[4],q[3];
ccx q[0],q[1],q[4];
```



Reasoning about quantum programs

Ideally, we would like a computer to know **what a quantum program does**

At the very least, it should know **when two programs are equivalent** for the purposes of

- Verification
- Compilation
- Optimization

```
%a1, %b1 = qs.CX %a0, %b0
%a2, %b2 = qs.CX %a1, %b1
%a3 = qs.H %a2
```

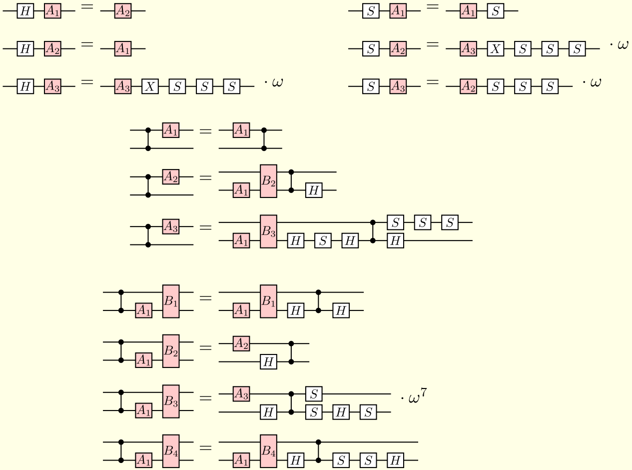
$\longrightarrow$

```


%a3 = qs.H %a0
```

Ittah et al., Enabling Dataflow Optimization for Quantum Programs. ACM Trans. Quant. Comput. (2022).

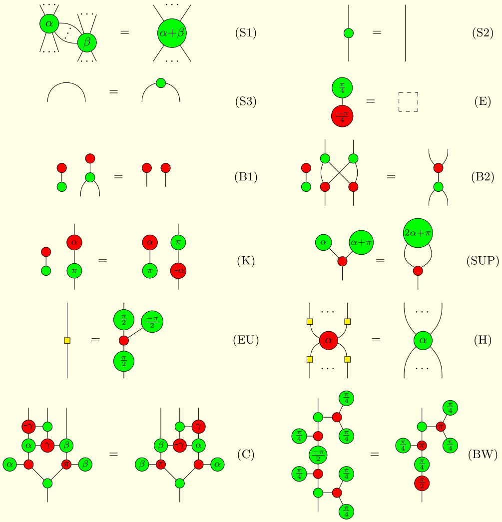
Circuit reasoning

$$\omega^8 = 1$$



Selinger, Generators and relations for n-qubit Clifford operators. LMCS 2015.

ZX reasoning



Jeandel, Perdrix and Vilmart, A Complete Axiomatisation of the ZX-Calculus for Clifford+T Quantum Mechanics. LICS 2018.

Models of classical computation

Combinatory logic

- Syntax: Point-free compositions of operators over a basis (e.g. **circuits**)
- Reasoning: Equational (e.g. by **re-write rules**)

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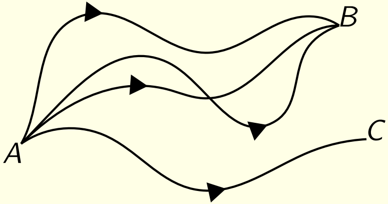
Is there a more computational model of QC?

Enter the symbolic sum-over-paths



A **symbolic representation** of discrete path integrals which is:

- Efficiently computable
- Universal for qubit quantum mechanics
- Re-writing has a **computational** interpretation as reducing or contracting sets of interfering paths
  - Highly automatable!
  - Implemented in FEYNMAN (`https://github.com/meamy/feynman`)

Amy, Towards large-scale functional verification of universal quantum circuits. QPL 2018.

Overview

1. The sum-over-paths
2. Reasoning with symbolic sums
3. Applications

The sum-over-paths

The linear-algebraic view

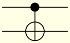
A (pure) state of n qubits is a unit vector in $\mathbb{C}^{2^n}$ which can be described as superposition of classical states

$$|\psi\rangle = \sum\nolimits_{\mathsf{x}\in\mathbb{Z}_2^n} \alpha_{\mathsf{x}}|\mathsf{x}\rangle, \qquad \mathsf{x} \in \{0,1\}^n = \mathbb{Z}_2^n$$

Computations change the state by applying unitary transformations to them

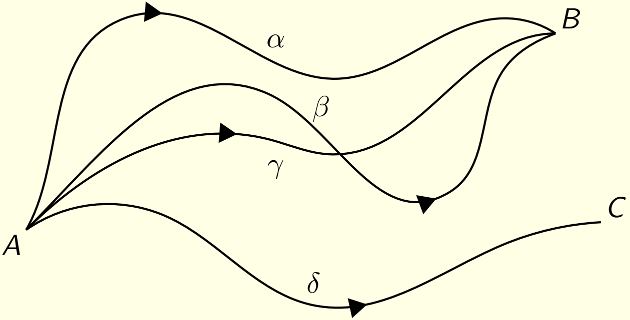
$$X = -\boxed{X}- = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad H = -\boxed{H}- = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$S = -\boxed{S}- = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \quad T = -\boxed{T}- = \begin{bmatrix} 1 & 0 \\ 0 & \omega = e^{i\frac{\pi}{4}} \end{bmatrix}$$

CNOT =  $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

The path integral view

The amplitude of a classical state is the **sum-over-all-paths leading to it**

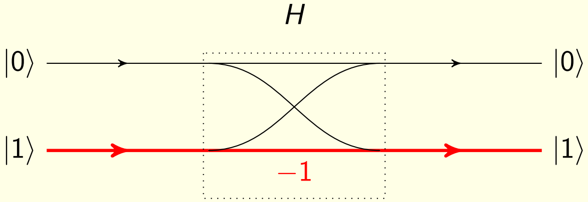


Computations change the state by sending classical states along various paths to new states

$$A \mapsto (\alpha + \beta + \gamma)B + \delta C$$

Gates as symbolic sums

The hadamard gate H **branches** on a **classical** value in superposition with equal weight $\frac{1}{\sqrt{2}}$ and varying phase



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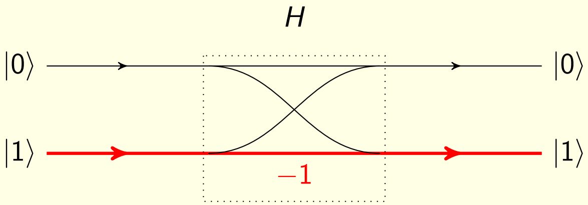
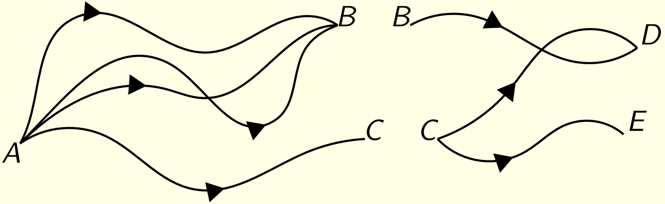


We can write this action symbolically as a **sum-over-paths**:

$$H : |x\rangle \mapsto \frac{1}{\sqrt{2}} \sum_{y \in \mathbb{Z}_2} (-1)^{xy} |y\rangle \text{ for } x \in \mathbb{Z}_2$$

Composition

Computations can be composed by composing paths through the same intermediate states



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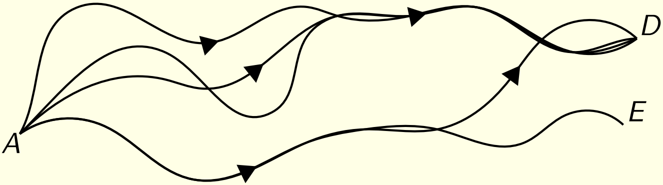

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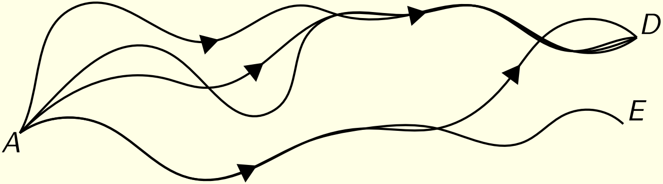

Symbolically, corresponds to an **encoding of matrix multiplication**

$$
\begin{aligned}
HH : |x\rangle &\mapsto \frac{1}{\sqrt{2}} \sum\nolimits_{y \in \mathbb{Z}_2} (-1)^{xy} H|y\rangle \\
&\mapsto \frac{1}{\sqrt{2}} \sum\nolimits_{y \in \mathbb{Z}_2} (-1)^{xy} \left(\frac{1}{\sqrt{2}} \sum\nolimits_{z \in \mathbb{Z}_2} (-1)^{yz} |z\rangle \right) \\
&\mapsto \frac{1}{2} \sum\nolimits_{y,z} (-1)^{xy+yz} |z\rangle
\end{aligned}
$$

Historical complexity applications

$$H : |x\rangle \mapsto \frac{1}{\sqrt{2}} \sum_{y \in \mathbb{Z}_2} (-1)^{xy} |y\rangle \qquad CCZ : |xyz\rangle \mapsto (-1)^{xyz} |xyz\rangle$$

The general form of a path sum over $\{H, CCZ\}$

$$|\mathrm{x}\rangle \mapsto \frac{1}{\sqrt{2}^{k+n}} \sum_{\mathrm{y} \in \mathbb{Z}_2^k} \sum_{\mathrm{x}' \in \mathbb{Z}_2^n} (-1)^{f(\mathrm{x},\mathrm{y},\mathrm{x}')} |\mathrm{x}'\rangle$$

where $\deg(f) \leq 3$.

Theorem (Ehrenfeucht & Karpinski)

Counting the 0's of $f \in \mathbb{Z}_2[x_1, \ldots]$ with degree ≥ 3 is #P-hard

Corollary: $\mathbf{BQP} \subseteq \mathbf{P}^{\#\mathbf{P}}$

Montanaro, Quantum circuits and low-degree polynomials over $\mathbb{Z}_2$. J Phys A: Math Theor, 2017.

Historical complexity applications

$$H : |x\rangle \mapsto \frac{1}{\sqrt{2}} \sum_{y \in \mathbb{Z}_2} (-1)^{xy} |y\rangle \qquad CZ : |xy\rangle \mapsto (-1)^{xy} |xy\rangle$$

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where $\deg(f) \leq 2$.

Theorem (Ehrenfeucht & Karpinski)

Counting the 0*'s of* $f \in \mathbb{Z}_2[x_1, \ldots]$ *with degree* ≤ 2 *is in P*

Corollary: $\{H, CZ\}$ **can be simulated in polynomial time**

Montanaro, Quantum circuits and low-degree polynomials over $\mathbb{Z}_2$. J Phys A: Math Theor, 2017.

Formalizing the sum-over-paths

Definition

A **(balanced) sum-over-paths** from $\mathbb{C}^{2^n}$ to $\mathbb{C}^{2^m}$ is a map

$$|x_1 \cdots x_n\rangle \mapsto \mathcal{N} \sum_{y_1,\ldots,y_k \in \mathbb{Z}_2} e^{2\pi i P(\mathrm{x},\mathrm{y})} |f_1(\mathrm{x},\mathrm{y}) \cdots f_m(\mathrm{x},\mathrm{y})\rangle$$

defined by

- a scalar $\mathcal{N} \in \mathbb{C}$,
- a **phase polynomial** $P \in \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots y_k]$, and
- outputs $f_i \in \mathbb{Z}_2[x_1, \ldots, x_n, y_1, \ldots y_k]$

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We can also consider **unbalanced** sums of the form

$$|x_1 \cdots x_n\rangle \mapsto \mathcal{N} \sum_{y_1,\ldots,y_k \in \mathbb{Z}_2} \alpha_1^{P_1(\mathrm{x},\mathrm{y})} \alpha_2^{P_2(\mathrm{x},\mathrm{y})} \cdots |f_1(\mathrm{x},\mathrm{y}) \cdots f_m(\mathrm{x},\mathrm{y})\rangle$$

The sum-over-paths as a model for circuits

Balanced sums form a **symmetric monoidal category**, so we can define the circuit SOP $[\![C]\!]$ **compositionally** by giving interpretations of each basis gate:

$$[\![H]\!] = |x\rangle \mapsto \frac{1}{\sqrt{2}}\sum\nolimits_y (-1)^{xy}|y\rangle$$
$$[\![T]\!] = |x\rangle \mapsto \omega^x|x\rangle$$
$$[\![\mathrm{CNOT}]\!] = |x_1x_2\rangle \mapsto |x_1(x_1 \oplus x_2)\rangle$$

$$[\![U_2U_1]\!] = [\![U_2]\!] \circ [\![U_1]\!]$$
$$[\![U_1 \otimes U_2]\!] = [\![U_1]\!] \otimes [\![U_2]\!]$$

Proposition

For any fixed k, the discrete path integral of a circuit over Clifford $+ R_k := diag(1, e^{2\pi i/2^k})$ is poly-time and poly-space computable

Amy, Towards large-scale functional verification of universal quantum circuits. QPL 2018.

Universality

Balanced sums can be further equipped with the structure of a **dagger compact category** via a unit, counit, and dagger

$$[\![\eta]\!] = \sum_y |yy\rangle$$

$$[\![\epsilon]\!] = |x_1 x_2\rangle \mapsto \sum_y (-1)^{x_1 y + y x_2}$$

$$[\![U^\dagger]\!] = [\![U]\!]^\dagger$$

Theorem

Any linear operator between even-power dimensional complex vector spaces can be represented as a balanced sum-over-paths

- $\langle\psi| := |\psi\rangle^\dagger$
- $tr_A(U) := (\epsilon \otimes I) \circ (I \otimes U) \circ (\eta \otimes I)$
- $meas := (SWAP \otimes \epsilon) \circ (I \otimes \chi_{meas} \otimes I) \circ (\eta \otimes SWAP)$

Vilmart, The Structure of Sum-Over-Paths, its Consequences, and Completeness for Clifford. FoSSaCS 2021.

Reasoning with symbolic sums

Rewriting sums

The sum-over-paths representation **encodes** matrix multiplication **symbolically**

$$[\![I]\!] = |x\rangle \mapsto |x\rangle$$
$$[\![HH]\!] = |x\rangle \mapsto \frac{1}{2}\sum_{y,z\in\mathbb{Z}_2}(-1)^{xy+yz}|z\rangle$$

This allows the sum to be **simplified** without explicit evaluation!

Interference patterns

To see how the *HH* sum

$$|x\rangle \mapsto \frac{1}{2} \sum_{y,z \in \mathbb{Z}_2} (-1)^{xy+yz} |z\rangle$$

interferes, we can expand it out.

$|x\rangle$ ———

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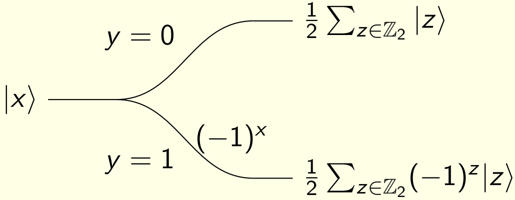

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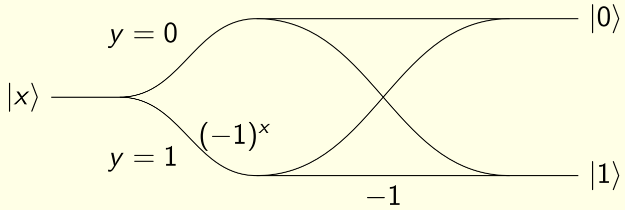
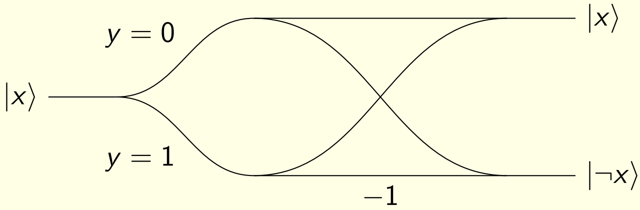


Interference patterns

If we sum over $z \in \{x, \neg x\} = \mathbb{Z}_2$ instead,

$$|x\rangle \mapsto \frac{1}{2} \sum_{y \in \mathbb{Z}_2, z \in \{x, \neg x\}} (-1)^{xy+yz} |z\rangle$$

we get a simple pattern



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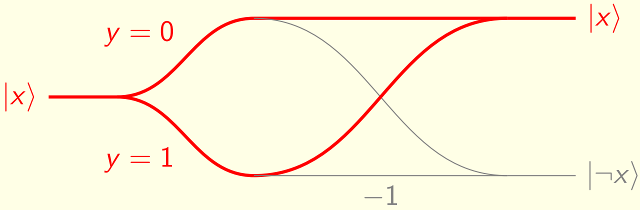

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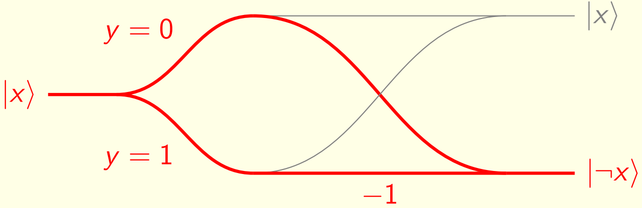


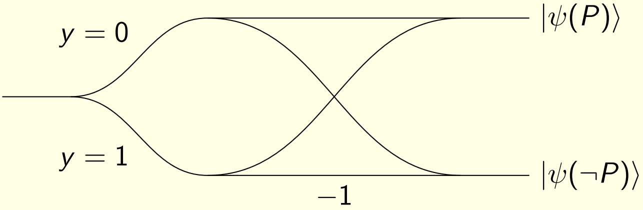
Generalization

Lemma

For any Boolean-valued expression P

$$\sum_{y,z}(-1)^{zy+yP}|\psi(z)\rangle = 2|\psi(P)\rangle$$

In particular **only the paths where $z = P$ survive**



Reduction rules for sum-over-paths

$$\sum_y |\psi\rangle \quad\longleftrightarrow\quad 2|\psi\rangle \qquad [E]$$

$$\sum_{y,z}(-1)^{zy+yP}|\psi(z)\rangle \quad\longleftrightarrow\quad 2|\psi(P)\rangle \qquad [H]$$

$$\sum_y i^y(-1)^{yP}|\psi\rangle \quad\longleftrightarrow\quad \sqrt{2}\omega i^{3P}|\psi\rangle \qquad [\omega]$$

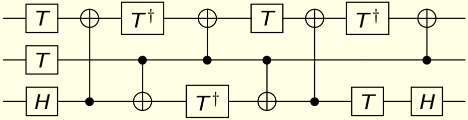
$$\sum_{y,z}\alpha^{xP(y)}\beta^{\neg xQ(z)}|\psi\rangle \quad\longleftrightarrow\quad \sum_y \alpha^{xP(y)}\beta^{\neg xQ(y)}|\psi\rangle \qquad [O]$$

$$\sum_y(\alpha^y\beta^{\neg y})^P|\psi\rangle \quad\longleftrightarrow\quad 2(\frac{\alpha+\beta}{2})^P|\psi\rangle \qquad [A]$$

- **[E], [H], [ω]** complete (and poly-time) for **Clifford** circuits
- **[E], [H], [ω], [O], [A]** complete for arbitrary linear operators
- Vilmart 2022: a complete re-write system for Clifford+R_k that **stays in the balanced fragment**

Reasoning symbolically

What computation does this circuit perform?



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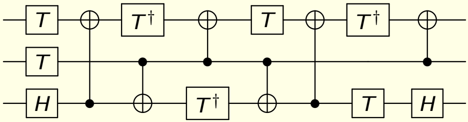

$$|x_1 x_2 x_3\rangle \mapsto \frac{1}{2} \sum_{y_1, y_2 \in \mathbb{Z}_2} (-1)^{x_3 y_1 + x_1 x_2 y_1 + y_1 y_2} |x_1 x_2 y_2\rangle$$

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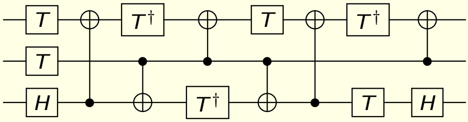

$$|x_1x_2x_3\rangle \mapsto \frac{1}{2}\sum_{y_1,y_2\in\mathbb{Z}_2}(-1)^{x_3y_1+x_1x_2y_1+y_1y_2}|x_1x_2y_2\rangle$$

$$\mapsto \frac{1}{2}\sum_{y_1,y_2\in\mathbb{Z}_2}(-1)^{y_2y_1+y_1(x_3+x_1x_2)}|x_1x_2y_2\rangle$$

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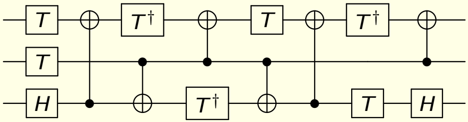

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$$\mapsto \frac{1}{2}\sum_{y_1,y_2\in\mathbb{Z}_2}(-1)^{y_2y_1+y_1(x_3+x_1x_2)}|x_1x_2y_2\rangle$$

$$\mapsto |x_1x_2(x_3\oplus x_1x_2)\rangle \qquad [\mathrm{H},\ y_2\leftarrow x_3\oplus x_1x_2]$$

Applications: Verification & simulation

Why verify?

- **High** degree of uncertainty about the correctness of estimates
- Bugs!

[a]Our simulation found an error in the circuit optimized by T-par. Specifically, the circuit maps $|1024\rangle \mapsto \frac{|1025\rangle+|1030\rangle+|1161\rangle+|1166\rangle}{2}$ whereas it is supposed to perform the mapping $|1024\rangle \mapsto |1088\rangle$.

[b]Note that our software reduced the T-count of the original pre-optimization circuit used by T-par to 0. It turned out that the circuit used by T-par is incorrect. In our optimization reported in this table, we used the correct original circuit [41, Figure 5].

qcla-mod$_7$ [27]	26	413	235[a]	NRSCM	237	**216**
rc-adder$_6$ [27]	14	77	47	RM$_{m\&r}$	47	47
vbe-adder$_3$ [27]	10	70	24	Tpar	24	24

Table 1: Benchmark circuits. The columns n and T contain the amount of qubits and T gates in the original circuit. *Best prev.* is the previous best-known ancilla-free T-count for that circuit and *Method* specifies which method was used: *RM$_m$* and *RM$_r$* refer to the *maximum* and *recursive* Reed-Muller decoder of Ref. [8], *Tpar* is Ref. [6], *TODD* is Ref. [21] and *NRSCM* refers to Ref. [28]. *PyZX* and *PyZX + TODD* specify the T-counts produced by the procedures outlined in the Methods section. Numbers shown in bold are better than previous best, and italics are worse. The superscript (a) indicates an error in a previously reported T-count. The error was found using Amy's Feynman tool [4].

[4]It was reported in [2] that 4 separate optimized benchmark circuits (CSLA-MUX_3, Adder_8, Mod-Mult_55 and GF(2^{32})) provably contained errors. These errors have since been fixed and these circuits now pass verification.

Nam, Ross, Su, Childs, Maslov, *Automated optimization of large quantum circuits with continuous parameters*. npj:Quantum Information 4, 23 (2018)

Amy, Azimzadeh, Mosca, *On the CNOT-complexity of CNOT-PHASE circuits*. Quantum Science and Technology 4(1), 2018

van de Wetering, Kissinger, *Reducing T-count with the ZX-calculus*. Phys. Rev. A 102, 022406 (2020)

Automated verification in practice

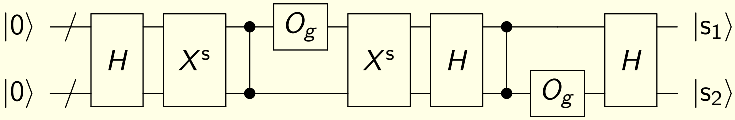
Through **purely automated re-writing**

- Circuits verified against **logical specifications** [A18] (below)
- Optimization results verified in academic papers
  - [AAM18, dBBW19, dBBW20, vdWK20, AR21]
  - Bugs also found in these!
- State & T gate teleportation channels verified
- Measurement-assisted uncomputation circuits verified for up to 128 qubits in [AR12]

Circuit	Qubits	Variables	Gates	Time (s)
Toffoli_{50}	97	190	1520	1.078
Toffoli_{100}	**197**	390	3120	**5.346**
Maslov_{50}	74	192	865	0.759
Maslov_{100}	149	392	1765	3.937
Adder_{8}	40	56	530	0.142
Adder_{16}	80	120	1130	26.151
QFT_{16}	16	16	616	1.250
QFT_{31}	31	31	2356	16.929

Hidden shift algorithm

Simulation of **the entire output distribution** for the popular Maiorana-McFarland Hidden Shift benchmark circuits via only automated re-writing



- **5s** vs **28h** for stabilizer decompositions[1]
- **1m** on a **tablet computer** for 1400 T-count, $> 10,000$ gate instances used recently in graphical simulation[2]

[1]S. Bravyi et al., *Simulation of quantum circuits by low-rank stabilizer decompositions.* Quantum 3, 181 (2019).

[2]A. Kissinger, J. van de Wetering, R. Vilmart, Classical simulation of quantum circuits with partial and graphical stabiliser decompositions. QPL 2022.

Applications:
Synthesis & Optimization

Clifford circuits

Over $\{H, CNOT, S\}$, path sums have the form

$$|x\rangle \mapsto \frac{1}{\sqrt{2^m}} \sum_{\mathrm{y} \in \mathbb{Z}_2^m} i^{L(\mathrm{x},\mathrm{y})} (-1)^{Q(\mathrm{x},\mathrm{y})} |f(\mathrm{x}, \mathrm{y})\rangle$$

where L is linear, Q is pure quadratic, and f is affine.

Amy, Bennett-Gibbs, Ross, Symbolic synthesis of Clifford circuits and beyond. QPL 2022.

Clifford circuits

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where L is linear, Q is pure quadratic, and f is affine.

Proposition (Affine normal form)

Any Clifford circuit can be written, up to a re-ordering of the qubits, using the re-write rules [E], [H], [ω] in polynomial time as

$$|\mathrm{x}\rangle \mapsto \frac{\omega^l}{\sqrt{2^k}} \sum_{\mathrm{y} \in \mathbb{Z}_2^k} i^{L(\mathrm{x},\mathrm{y})}(-1)^{Q(\mathrm{x},\mathrm{y})}|\mathrm{y}\rangle \otimes |f(\mathrm{x},\mathrm{y})\rangle$$

Corollary: **Efficient simulation of Clifford circuits**

Amy, Bennett-Gibbs, Ross, Symbolic synthesis of Clifford circuits and beyond. QPL 2022.

Decomposition into linear operators

The affine normal form

$$|\mathrm{x}\rangle \mapsto \frac{\omega^l}{\sqrt{2^k}} \sum_{\mathrm{y}\in\mathbb{Z}_2^k} i^{L(\mathrm{x},\mathrm{y})}(-1)^{Q(\mathrm{x},\mathrm{y})}|\mathrm{y}\rangle \otimes |f(\mathrm{x},\mathrm{y})\rangle$$

decomposes into the following sequence of linear operators

$$\begin{aligned}
|\mathrm{x}\rangle &\mapsto \omega^l i^{L_x(\mathrm{x})}(-1)^{Q_x(\mathrm{x})}|\mathrm{x}\rangle && \{\mathrm{S}, \mathrm{CZ}\} \\
|\mathrm{x}\rangle &\mapsto |R(\mathrm{x})\rangle|f_x(\mathrm{x})\rangle && \{\mathrm{CNOT}\} \\
|R(\mathrm{x})\rangle|f_x(\mathrm{x})\rangle &\mapsto \frac{1}{\sqrt{2^k}} \sum_{\mathrm{y}\in\mathbb{Z}_2^k} (-1)^{\sum_i y_i R_i(\mathrm{x})}|\mathrm{y}\rangle|f_x(\mathrm{x})\rangle && \{\mathrm{H}\} \\
|\mathrm{y}\rangle|f_x(\mathrm{x})\rangle &\mapsto |\mathrm{y}\rangle|f_x(\mathrm{x}) + f_y(\mathrm{y}) + \mathrm{b}\rangle && \{\mathrm{X}, \mathrm{CNOT}\} \\
|\mathrm{y}\rangle|f(\mathrm{x},\mathrm{y})\rangle &\mapsto i^{L_y(\mathrm{y})}(-1)^{Q_y(\mathrm{y})}|\mathrm{y}\rangle|f(\mathrm{x},\mathrm{y})\rangle && \{\mathrm{S}, \mathrm{CZ}\}
\end{aligned}$$

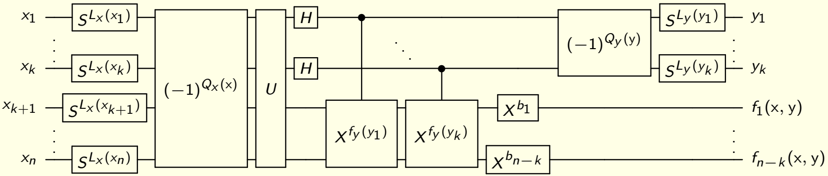
Amy, Bennett-Gibbs, Ross, Symbolic synthesis of Clifford circuits and beyond. QPL 2022.

A constructive proof of the Bruhat decomposition

Theorem

Any Clifford operator can be synthesized in polynomial time over $\{\mathrm{CNOT}, \mathrm{X}, \mathrm{CZ}, \mathrm{S}, \mathrm{H}\}$ *as an 8-stage circuit of the form*

$$\mathrm{S} \cdot \mathrm{CZ} \cdot \mathrm{CNOT} \cdot \mathrm{H} \cdot \mathrm{CNOT} \cdot \mathrm{X} \cdot \mathrm{CZ} \cdot \mathrm{S}$$



Maslov, Roetteler, Shorter stabilizer circuits via Bruhat decomposition and quantum circuit transformations. IEEE TIT 2018.

Amy, Bennett-Gibbs, Ross, Symbolic synthesis of Clifford circuits and beyond. QPL 2022.

Synthesizing more general circuits

Can we synthesize non-Clifford operators?

Amy, Bennett-Gibbs, Ross, Symbolic synthesis of Clifford circuits and beyond. QPL 2022.

Synthesizing more general circuits

Can we synthesize non-Clifford operators?

By **inverting** the sum-over-paths actions, we get an augmented re-write system with side effects

$$\Lambda_k(X) : |\mathsf{x}\rangle|y \oplus \prod_i x_i\rangle \mapsto |\mathsf{x}\rangle|y\rangle$$

$$\Lambda_k(\theta) : e^{2\pi i\theta \prod_i x_i}|\mathsf{x}\rangle \mapsto |\mathsf{x}\rangle$$

$$H : \frac{1}{\sqrt{2}} \sum_{x' \in \mathbb{Z}_2} (-1)^{xx'}|x'\rangle \mapsto |x\rangle$$

Synthesize by reducing to the identity!

Amy, Bennett-Gibbs, Ross, Symbolic synthesis of Clifford circuits and beyond. QPL 2022.

Synthesizing the QFT

The n-bit **QFT** is given by the following matrix

$$\frac{1}{\sqrt{2^n}}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_{2^n} & \omega_{2^n}^2 & \omega_{2^n}^3 & \cdots & \omega_{2^n}^{2^n-1} \\ 1 & \omega_{2^n}^2 & \omega_{2^n}^4 & \omega_{2^n}^6 & \cdots & \omega_{2^n}^{2(2^n-1)} \\ 1 & \omega_{2^n}^3 & \omega_{2^n}^6 & \omega_{2^n}^9 & \cdots & \omega_{2^n}^{3(2^n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_{2^n}^{2^n-1} & \omega_{2^n}^{2(2^n-1)} & \omega_{2^n}^{3(2^n-1)} & \cdots & \omega_{2^n}^{(2^n-1)(2^n-1)} \end{bmatrix}$$

As a sum-over-paths,

$$QFT_n : |\mathsf{x}\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{\mathsf{y}\in\mathbb{Z}_2^n} \omega_{2^n}^{\mathsf{x}\cdot\mathsf{y}} |\mathsf{y}\rangle$$

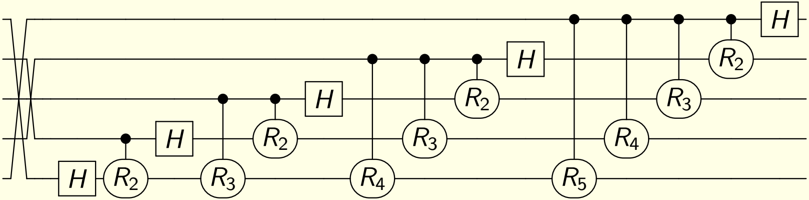
QFT_3 derivation

Completely automated circuit derivation

$$
\begin{aligned}
QFT_3|x_1x_2x_3\rangle &= \frac{1}{\sqrt{2^3}} \sum\nolimits_{y_1,y_2,y_3} \omega^{x_3y_3} i^{x_3y_2+x_2y_3} (-1)^{x_3y_1+x_2y_2+x_1y_3} |y_1y_2y_3\rangle \\
&\xrightarrow{H_1} \frac{1}{\sqrt{2^2}} \sum\nolimits_{y_1,y_2} \omega^{x_3y_3} i^{x_3y_2+x_2y_3} (-1)^{x_2y_2+x_1y_3} |x_3y_2y_3\rangle \\
&\xrightarrow{cS^\dagger_{1,2}cT^\dagger_{1,3}} \frac{1}{\sqrt{2^2}} \sum\nolimits_{y_2,y_3} i^{x_2y_3} (-1)^{x_2y_2+x_1y_3} |x_3y_2y_3\rangle \\
&\xrightarrow{H_2} \frac{1}{\sqrt{2}} \sum\nolimits_{y_3} i^{x_2y_3} (-1)^{x_1y_3} |x_3x_2y_3\rangle \\
&\xrightarrow{cS^\dagger_{2,3}} \frac{1}{\sqrt{2}} \sum\nolimits_{y_3} (-1)^{x_1y_3} |x_3x_2y_3\rangle \\
&\xrightarrow{H_3} |x_3x_2x_1\rangle
\end{aligned}
$$

QFT, synthesized

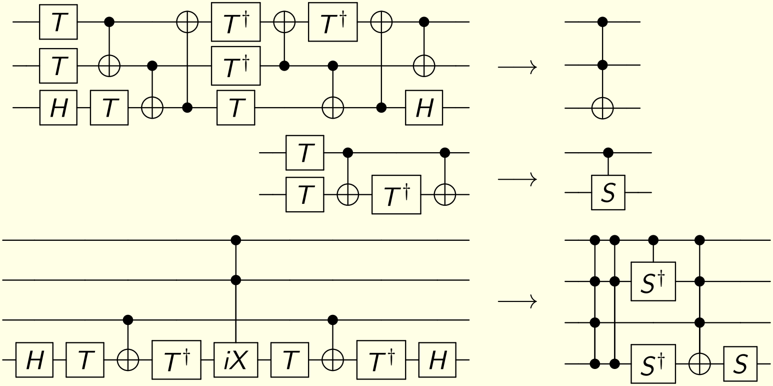
Compiled with FEYNMAN:



https://github.com/meamy/feynman

Decompilation

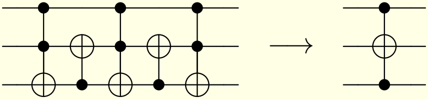
Sum-over-paths synthesis can be used to **decompile** from a low-level gate set to a high-level one

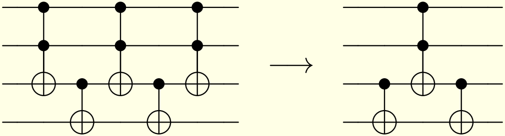


Reveals useful semantic information about a circuit!

Circuit optimization

(Future work) Can re-synthesis be used to optimize quantum circuits?





Conclusion

In this talk...

- Symbolic sums as a representation for quantum circuits, channels, and generally mechanics
- Reasoning with symbolic sums
- Applications to simulation, verification, synthesis and circuit optimization

Thank you!