Complete equational theories for the sum-over-paths with unbalanced amplitudes

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The sum-over-paths

In the path integral formulation of QM, state evolution is viewed as a sum over the paths taken in superposition

$$\Psi|i\rangle = \sum_{\pi \in \Pi} \psi(\pi)|f(\pi)\rangle$$

With certain representations of the sum-over-paths,

- supports efficient composition
- can be re-written symbolically to reduce interfering paths
- highly amenable to automated **simplification**, **verification** & **simulation** of some circuits

Symbolic representation of balanced sums

Recall: A **balanced** path sum is a symbolic expression of a linear operator $\Psi : \mathbb{C}^{2^m} \to \mathbb{C}^{2^n}$ as a sum of the form

$$\Psi|\vec{x}\rangle = |\Psi(\vec{x})\rangle = \mathcal{N} \sum_{\vec{y}\in\mathbb{Z}_2^k} e^{2\pi i P(\vec{x},\vec{y})}|f(\vec{x},\vec{y})\rangle,$$

- $\mathcal{N} \in \mathbb{C} \setminus \{0\}$ is a normalization factor,
- $P : \mathbb{Z}_2^m \times \mathbb{Z}_2^k \to \mathbb{R}$ is a real-valued multilinear polynomial, and
- $f : \mathbb{Z}_2^m \times \mathbb{Z}_2^k \to \mathbb{Z}_2^n$ is system of n Boolean polynomials

Notation

$|\Psi(x, y, ...)\rangle$ *refers to a path sum expression with distinguished free variables* $x, y, ...$

Examples

Phase & reversible gates:

$$S|x\rangle = i^x|x\rangle, \qquad T|x\rangle = e^{\frac{i\pi}{4}x}|x\rangle$$

$$X|x\rangle = |1 \oplus x\rangle \qquad CNOT|x\rangle|y\rangle = |x\rangle|x \oplus y\rangle$$

Branching gates:

$$H|x\rangle = \frac{1}{\sqrt{2}}\sum_y (-1)^{xy}|y\rangle$$

Cups and caps:

$$\subset = \sum_y |y\rangle|y\rangle, \qquad \supset |x_1\rangle|x_2\rangle = \frac{1}{2}\sum_y (-1)^{y(x_1+x_2)}$$

Equational reasoning

If $|\Psi\rangle$ is a path sum where $y \notin FV(\Psi)$ and f is a Boolean expression such that $x, y \notin FV(f)$, then the following hold:

$$\sum_{y \in \mathbb{Z}_2} |\Psi\rangle \equiv_{\text{Cliff}} 2|\Psi\rangle \tag{E}$$

$$\sum_{x,y \in \mathbb{Z}_2} (-1)^{y(x+f)} |\Psi(x)\rangle \equiv_{\text{Cliff}} 2|\Psi(f)\rangle \tag{H}$$

$$\sum_{y \in \mathbb{Z}_2} i^y (-1)^{yf} |\Psi\rangle \equiv_{\text{Cliff}} \omega\sqrt{2}(-i)^{\overline{f}} |\Psi\rangle \tag{ω}$$

Theorem

$\equiv_{Cliff}$ *is complete for Stabilizer operations*

Completeness for Toffoli+Hadamard

$$\sum_{y\in\mathbb{Z}_2}\sum_{x\in\mathbb{Z}_2}(-1)^{y(x\cdot g+g\cdot f+1)}|\Psi(x)\rangle \equiv_{\mathrm{TH}} \sum_{y\in\mathbb{Z}_2}(-1)^{y(g+1)}|\Psi(1+f)\rangle \quad \text{(Hgen)}$$

$$\sum_{y\in\mathbb{Z}_2}\sum_{x\in\mathbb{Z}_2}(-1)^{y\cdot f+x\cdot g}|\Psi\rangle \equiv_{\mathrm{TH}} 2\sum_{y\in\mathbb{Z}_2}(-1)^{y(f+g+f\cdot g)}|\Psi\rangle \quad \text{(Hrel)}$$

$$\sum_{y\in\mathbb{Z}_2}\sum_{x\in\mathbb{Z}_2}(-1)^{y}|\Psi\rangle \equiv_{\mathrm{TH}} 0|\Psi\rangle \quad \text{(Z)}$$

Theorem (Vilmart 2022)

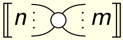
$\equiv_{TH+Cliff}$ *is complete for Toffoli-Hadamard and by extension Clifford*$+R_Z(2\pi/2^k)$

Renaud Vilmart, Completeness of Sum-Over-Paths for Toffoli-Hadamard and the Dyadic Fragments of Quantum Computation. CSL 2023.

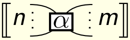
Goal: can we translate the full ZH-calculus to a complete equational theory for arbitrary path sums?

From ZH to SOP

Recall: The ZH-calculus[1] is generated by



$$= \sum_{x} |xx \cdots x\rangle\langle xx \cdots x|$$



$$= \sum_{\vec{x},\vec{y}} \alpha^{x_1 \cdots x_n y_1 \cdots y_m} |\vec{y}\rangle\langle\vec{x}|$$

[1] Miriam Backens and Aleks Kissinger, ZH: A Complete Graphical Calculus for Quantum Computations Involving Classical Non-linearity. QPL 2018.

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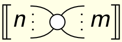

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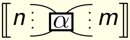
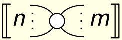
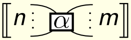


$$= \sum_{\vec{x},\vec{y}} \alpha^{x_1 \cdots x_n y_1 \cdots y_m} |\vec{y}\rangle\langle\vec{x}|$$

With a slight tweak we can write them in the style of path sums:



$$|\vec{x}\rangle = \sum_{y} \sum_{\vec{z}} 2^{-n}(-1)^{\sum_{i=1}^{n} z_i(x_i+y)} |yy \cdots y\rangle$$



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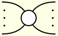
From ZH to SOP

Recall: The ZH-calculus[1] is generated by

$\llbracket n$  $m \rrbracket = \sum_{x} |xx\cdots x\rangle\langle xx\cdots x|$

$\llbracket n$  $m \rrbracket = \sum_{\vec{x},\vec{y}} \alpha^{x_1\cdots x_n y_1\cdots y_m} |\vec{y}\rangle\langle\vec{x}|$

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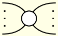
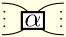
$\llbracket n$  $m \rrbracket\, |\vec{x}\rangle = \sum_{y}\sum_{\vec{z}} 2^{-n}(-1)^{\sum_{i=1}^{n} z_i(x_i+y)} |yy\cdots y\rangle$

$\llbracket n$  $m \rrbracket\, |\vec{x}\rangle = \sum_{\vec{y}} \alpha^{x_1\cdots x_n y_1\cdots y_m} |\vec{y}\rangle$

Problem: need a notion of unbalanced sums!

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Balanced vs Unbalanced

The controlled-H gate

$$CH = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

is **unbalanced** since non-zero entries differ in magnitude

Balanced representations of unbalanced operators like the controlled$-H$ gate are tricky to design and **simplify**!

$$CH|x_1x_2\rangle = \frac{1}{\sqrt{2}} \sum_{y \in \mathbb{Z}_2} \omega^{(1-x_1)(2y-1)}(-1)^{x_1x_2y}|x_1\rangle|(1 \oplus x_1)x_2 \oplus x_1y\rangle$$

An unbalanced sum-over-paths

An unbalanced sum-over-paths is an expression $|\Psi\rangle$ of the following language

$$f ::= 0 \mid 1 \mid x, y, z, ... \mid f_1 \cdot f_2 \mid f_1 \oplus f_2 \mid \neg f := 1 \oplus f$$
$$a ::= \alpha, \beta \in \mathcal{F} \mid a^f \mid a_1 a_2$$
$$|\Psi\rangle ::= \sum_{\vec{y}} a |f_1 \cdots f_n\rangle.$$

where variables $x, y, z, \ldots$ range over Booleans and $\mathcal{F}$ is a field.

Universality

Unbalanced sums subsume balanced sums and are trivially universal as any linear operator $\mathcal{F}^{2^n} \to \mathcal{F}^{2^m}$ may be uniquely written as

$$\Psi|\vec{x}\rangle = \sum_{\vec{y}} \alpha_{00\cdots 0}^{00\cdots 0=\vec{x}\vec{y}} \alpha_{00\cdots 1}^{00\cdots 1=\vec{x}\vec{y}} \cdots \alpha_{11\cdots 1}^{11\cdots 1=\vec{x}\vec{y}} |\vec{y}\rangle$$

where $\vec{x} = \vec{y} := \prod_i x_i \oplus y_i \oplus 1$.

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where $\vec{x} = \vec{y} := \prod_i x_i \oplus y_i \oplus 1$.

Definition (Normal form)

A **normal form** is a closed, unbalanced sum of the following form:

$$\sum_{\vec{x}} \alpha_{00\cdots 0}^{00\cdots 0 = \vec{x}} \alpha_{00\cdots 1}^{00\cdots 1 = \vec{x}} \cdots \alpha_{11\cdots 1}^{11\cdots 1 = \vec{x}} |\vec{x}\rangle$$

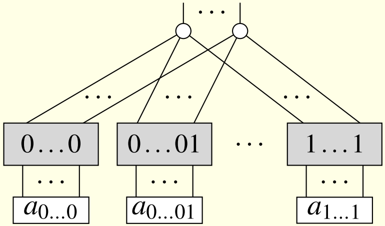
Normal forms

Unbalanced sums vs. ZH calculus

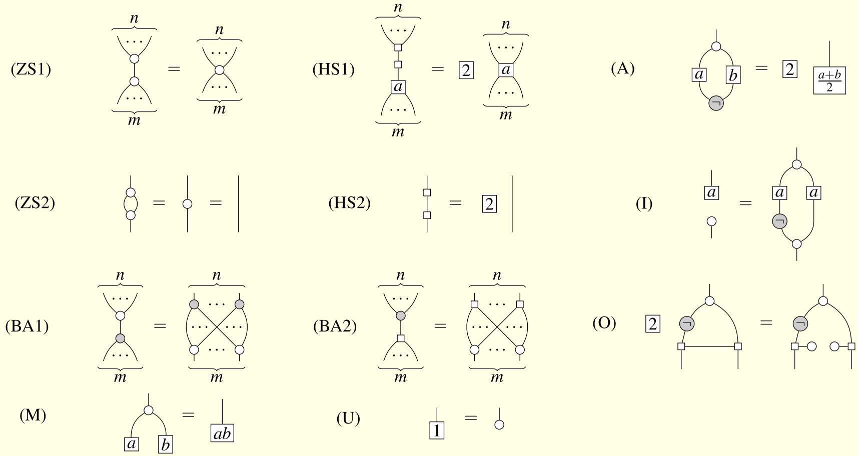
Unbalanced sums:

$$\sum_{\vec{x}} \alpha_{00\cdots 0}^{00\cdots 0=\vec{x}} \alpha_{00\cdots 1}^{00\cdots 1=\vec{x}} \cdots \alpha_{11\cdots 1}^{11\cdots 1=\vec{x}} |\vec{x}\rangle$$

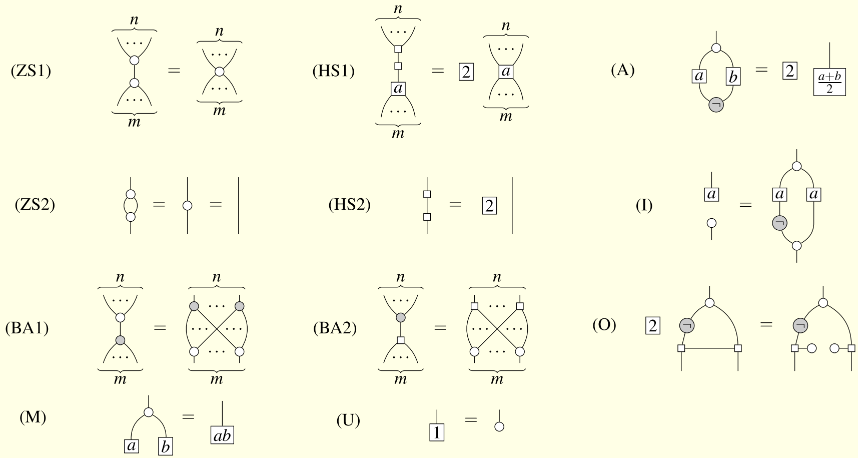
ZH-calculus:



Equational theory of the ZH-calculus



Equational theory of the ZH-calculus

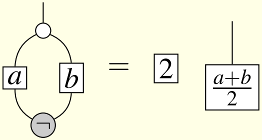


Only need analogues of (A) (I) and (O)

Formulating the average rule

ZH-calculus:

(A)



Formulating the average rule

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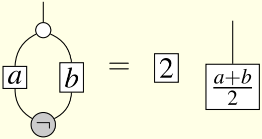
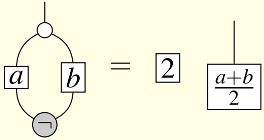


Unbalanced sums:

$$\sum_y (\alpha^y \beta^{\neg y})^x |\Psi\rangle = 2\left(\frac{\alpha+\beta}{2}\right)^x |\Psi\rangle$$

Formulating the average rule

ZH-calculus:

(A) 

Unbalanced sums:

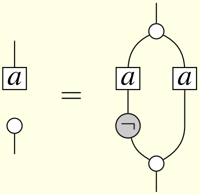
$$\sum_{y} (\alpha^{y} \beta^{\neg y})^{x} |\Psi\rangle = 2 \left(\frac{\alpha + \beta}{2} \right)^{x} |\Psi\rangle$$

In SOP-land, we can generalize x to arbitrary expressions:

$$\sum_{y} (\alpha^{y} \beta^{\neg y})^{f} |\Psi\rangle \equiv 2 \left(\frac{\alpha + \beta}{2} \right)^{f} |\Psi\rangle$$

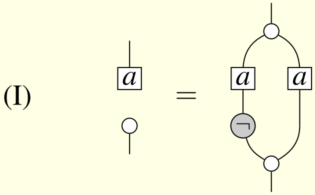
ZH-calculus:

(I)



Formulating the intro rule

ZH-calculus:

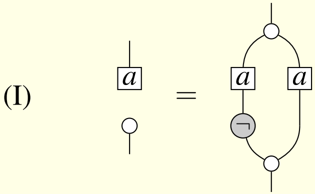


Unbalanced sums:

$$\sum_{y} a^{x} |y\rangle = \sum_{y} a^{xy} a^{x(\neg y)} |y\rangle$$

Formulating the intro rule

ZH-calculus:



Unbalanced sums:

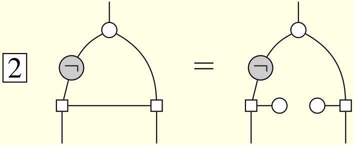
$$\sum_y a^x|y\rangle = \sum_y a^{xy}a^{x(\neg y)}|y\rangle$$

Generalized rule:

$$a \equiv a^y a^{\neg y}$$

Formulating the ortho rule

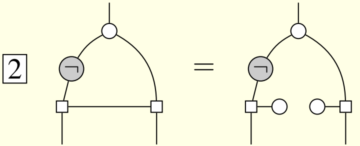
ZH-calculus:

(O) 

Formulating the ortho rule

ZH-calculus:

(O)

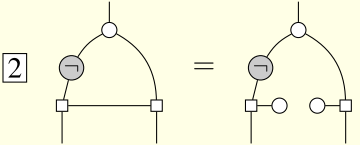


Unbalanced sums:

$$2 \sum_{w,y,z} (-1)^{(\neg x)wy} (-1)^{xwz} |yz\rangle = \sum_{w_1,w_2,y,z} (-1)^{(\neg x)w_1 y} (-1)^{x w_2 z} |yz\rangle$$

Formulating the ortho rule

ZH-calculus:

(O) 

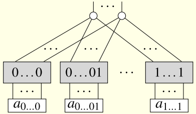
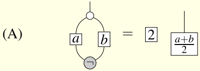
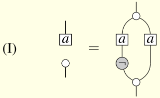
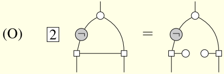
Unbalanced sums:

$$2 \sum_{w,y,z} (-1)^{(\neg x)wy} (-1)^{xwz} |yz\rangle = \sum_{w_1,w_2,y,z} (-1)^{(\neg x)w_1 y} (-1)^{xw_2 z} |yz\rangle$$

Thus a rule is born:

$$\sum_{y,z} a_1(y)^x a_2(z)^{\neg x} |\Psi(x)\rangle \equiv 2 \sum_{y} a_1(y)^x a_2(y)^{\neg x} |\Psi(x)\rangle$$

ZH vs. unbalanced sums

	ZH-calculus	unbalanced sums
Normal forms		$\sum_{\vec{x}} \alpha_{00\cdots0}^{00\cdots0=\vec{x}}\alpha_{00\cdots1}^{00\cdots1=\vec{x}}\cdots\alpha_{11\cdots1}^{11\cdots1=\vec{x}}\lvert\vec{x}\rangle$
(A) rule		$\sum_y (\alpha^y\beta^{\neg y})^f\lvert\Psi\rangle \equiv 2\left(\frac{\alpha+\beta}{2}\right)^f\lvert\Psi\rangle$
(I) rule		$a \equiv a^f a^{\neg f}$
(O) rule		$\sum_{y,z} a_1(y)^x a_2(z)^{\neg x}\lvert\Psi(x)\rangle \equiv 2\sum_y a_1(y)^x a_2(y)^{\neg x}\lvert\Psi(x)\rangle$
The rest		$\sum_{x,y}(-1)^{y(x\oplus f)}\lvert\Psi(x)\rangle \equiv 2\lvert\Psi(f)\rangle$

Rules of the unbalanced sum-over-paths

$$f \oplus 0 \equiv_{\mathcal{F}} f \qquad (f_1 \oplus f_2) \oplus f_3 \equiv_{\mathcal{F}} f_1 \oplus (f_2 \oplus f_3)$$

$$f \oplus f \equiv_{\mathcal{F}} 0 \qquad f_1 \oplus f_2 \equiv_{\mathcal{F}} f_2 \oplus f_1$$

$$f \cdot 1 \equiv_{\mathcal{F}} f \qquad (f_1 \cdot f_2) \cdot f_3 \equiv_{\mathcal{F}} f_1 \cdot (f_2 \cdot f_3)$$

$$f \cdot f \equiv_{\mathcal{F}} f \qquad f_1 \cdot f_2 \equiv_{\mathcal{F}} f_2 \cdot f_1$$

$$f_1 \cdot (f_2 \oplus f_3) \equiv_{\mathcal{F}} f_1 \cdot f_2 \oplus f_1 \cdot f_3$$

$$(a_1 \cdot a_2) \cdot a_3 \equiv_{\mathcal{F}} a_1 \cdot (a_2 \cdot a_3)$$

$$a_1 \cdot a_2 \equiv_{\mathcal{F}} a_2 \cdot a_1$$

$$a \cdot 1 \equiv_{\mathcal{F}} a$$

$$a^0 \equiv_{\mathcal{F}} 1 \equiv_{\mathcal{F}} 1^f$$

$$a^1 \equiv_{\mathcal{F}} a \equiv_{\mathcal{F}} a^f a^{\neg f}$$

$$a^{f_1 \oplus f_2} \equiv_{\mathcal{F}} a^{f_1} a^{f_2} (a^{-2})^{f_1 \cdot f_2}$$

$$a^{f_1 \cdot f_2} \equiv_{\mathcal{F}} (a^{f_1})^{f_2}$$

$$a_1^f a_2^f \equiv_{\mathcal{F}} (a_1 a_2)^f$$

$$\sum_{x,y} (-1)^{y(x \oplus f)} |\Psi(x)\rangle \equiv_{\mathcal{F}} 2|\Psi(f)\rangle \tag{H}$$

$$\sum_{y,z} a_1(y)^x a_2(z)^{\neg x} |\Psi(x)\rangle \equiv_{\mathcal{F}} 2 \sum_{y} a_1(y)^x a_2(y)^{\neg x} |\Psi(x)\rangle \tag{O}$$

$$\sum_{y} (\alpha^y \beta^{\neg y})^f |\Psi\rangle \equiv_{\mathcal{F}} 2 \left(\frac{\alpha + \beta}{2}\right)^f |\Psi\rangle \tag{A}$$

Completeness

Part 1: Normalization of amplitudes

Proposition ($\mathcal{F}$-expression normalization)

An $\mathcal{F}$-expression a can be brought into the form

$$\alpha_{00\cdots0}^{00\cdots0=\vec{x}}\alpha_{00\cdots1}^{00\cdots1=\vec{x}}\cdots\alpha_{11\cdots1}^{11\cdots1=\vec{x}}.$$

over the variables $\{x_i\} \supseteq FV(a)$ using the equations of $\equiv_{\mathcal{F}}$.

Theorem

$\equiv_{\mathcal{F}}$ is complete for unbalanced sums over any field $\mathcal{F}$.

Proof sketch:

1. Apply (H) to the sum $\sum_{\vec{y}} a|f_1 f_2 \cdots f_n\rangle$ to get $\sum_{\vec{x}} \sum_{\vec{y}} a'|\vec{x}\rangle$
2. Normalize a' to get
$$\sum_{\vec{x}} \sum_{\vec{y}} \alpha_{00\cdots00}^{00\cdots00=\vec{x}\vec{y}} \alpha_{00\cdots01}^{00\cdots01=\vec{x}\vec{y}} \cdots \alpha_{11\cdots11}^{11\cdots11=\vec{x}\vec{y}} |\vec{x}\rangle.$$
3. Sum amplitudes in pairs with (O) and (A) to remove each y_i:
$$\sum_{\vec{x}} \sum_{y} (\alpha_{00\cdots00}^{\neg y} \alpha_{00\cdots01}^{y})^{00\cdots0=\vec{x}} \cdots (\alpha_{11\cdots10}^{\neg y} \alpha_{11\cdots11}^{y})^{11\cdots1=\vec{x}} |\vec{x}\rangle$$
$$\equiv_{\mathcal{F}} \sum_{\vec{x}} \sum_{\vec{y}} \frac{1}{2^{2^n-1}} (\alpha_{00\cdots00}^{\neg y_1} \alpha_{00\cdots01}^{y_1})^{00\cdots0=\vec{x}} \cdots (\alpha_{11\cdots10}^{\neg y_n} \alpha_{11\cdots11}^{y_n})^{11\cdots1=\vec{x}} |\vec{x}\rangle$$
$$\equiv_{\mathcal{F}} \sum_{\vec{x}} \frac{2^{2^n}}{2^{2^n-1}} \left(\frac{\alpha_{00\cdots00} + \alpha_{00\cdots01}}{2}\right)^{00\cdots0=\vec{x}} \cdots \left(\frac{\alpha_{11\cdots10} + \alpha_{11\cdots11}}{2}\right)^{11\cdots1=\vec{x}} |\vec{x}\rangle$$

Completeness over rings

The $\oplus$-exponent equation $a^{f_1 \oplus f_2} \equiv a^{f_1} a^{f_2} (a^{-2})^{f_1 \cdot f_2}$ is not generally admissible over rings, e.g. $\mathbb{Z}[\frac{1}{\sqrt{2}}, i]$.

We can instead use $a^{f_1 \oplus f_2} \equiv a^{f_1} + a^{f_2} - (2a)^{f_1 \cdot f_2}$ if we allow amplitude expressions to be summed...

$$
\begin{aligned}
f &::= 0 \mid 1 \mid x, y, z, \ldots \mid f_1 \cdot f_2 \mid f_1 \oplus f_2 \\
r &::= \alpha, \beta \in \mathcal{R} \mid r^f \mid r_1 r_2 \mid r_1 + r_2 \\
|\Psi\rangle &::= \sum_{\vec{y}} r |f_1 \cdots f_n\rangle.
\end{aligned}
$$

where $\mathcal{R}$ is a commutative ring

Normalization of amplitude expressions

$$(r_1 + r_2) + r_3 \equiv r_1 + (r_2 + r_3)$$
$$r_1 + r_2 \equiv r_2 + r_1$$
$$r_1 + 0 \equiv r_1$$
$$r - r \equiv 0$$
$$(r_1 \cdot r_2) \cdot r_3 \equiv r_1 \cdot (r_2 \cdot r_3)$$
$$r_1 \cdot r_2 \equiv r_2 \cdot r_1$$
$$r \cdot 1 \equiv r$$
$$r_1 \cdot (r_2 + r_3) \equiv r_1 \cdot r_2 + r_1 \cdot r_3$$

$$r^0 \equiv 1 \equiv 1^f$$
$$r^1 \equiv r \equiv r^f r^{\neg f}$$
$$r^{f_1 \oplus f_2} \equiv r^{f_1} + r^{f_2} - (2r)^{f_1 \cdot f_2}$$
$$r^{f_1 \cdot f_2} \equiv (r^{f_1})^{f_2}$$
$$r_1^f r_2^f \equiv (r_1 r_2)^f$$
$$r_1^f r_2^{\neg f} \equiv r_1^f + r_2^{\neg f} - 1$$
$$r_1^f + r_2^f \equiv (r_1 + r_2)^f + 0^{\neg f}$$

Proposition

An amplitude expression r can be brought into the form

$$\alpha_{00\cdots0}^{00\cdots0=\vec{x}} \alpha_{00\cdots1}^{00\cdots1=\vec{x}} \ldots \alpha_{11\cdots1}^{11\cdots1=\vec{x}}$$

over the variables $\{x_i\} \supseteq FV(r)$ using the above equations (+ axioms of Boolean rings).

Completeness

$$\sum_{x,y}(-1)^{y(x\oplus f)}|\Psi(x)\rangle \equiv_{\mathcal{R}} 2|\Psi(f)\rangle \tag{H}$$

$$\sum_{y} r(y)|\Psi\rangle \equiv_{\mathcal{R}} (r(0)+r(1))|\Psi\rangle \tag{S}$$

Theorem

$\equiv_{\mathcal{R}}$ is complete for unbalanced sums over any field $\mathcal{R}$.

Proof sketch:

Use (H) to write the sum as $\sum_{\vec{y}}\sum_{\vec{z}} r(\vec{y},\vec{z})|\vec{z}\rangle$, then explicitly expand the sum over $\vec{y}$ with (S) and normalize the amplitude expression

More restrictive path sums

A more direct ZH analogue removes sums from both the **Amplitude and Boolean sub-languages**:

$$\begin{aligned} f &::= 0 \mid 1 \mid x, y, z, \ldots \mid f_1 \cdot f_2 \mid \cancel{f_1 \oplus f_2} \\ r &::= \alpha, \beta \in \mathcal{R} \mid r^f \mid r_1 r_2 \mid \cancel{r_1 + r_2} \\ |\Psi\rangle &::= \sum_{\vec{y}} r|f_1 \cdots f_n\rangle. \end{aligned}$$

In this case, we need rules explicitly handling distribution of Boolean sums, e.g. the following which corresponds to the BA2 rule of the ZH-calculus

$$\sum_{x,y}(-1)^{xy}(-1)^{yf_1}\cdots(-1)^{yf_k}(-1)^{xg}|\Psi\rangle \equiv 2(-1)^{f_1 g}\cdots(-1)^{f_k g}|\Psi\rangle,$$

Conclusion

So what did we actually do?

- Defined a notion of unbalanced path sums
- Gave complete equational theories over Rings & Fields

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Takeaway...

Symbolic algebra gives a lot of flexibility in the design and operation of formal methods for QC

Thank you!