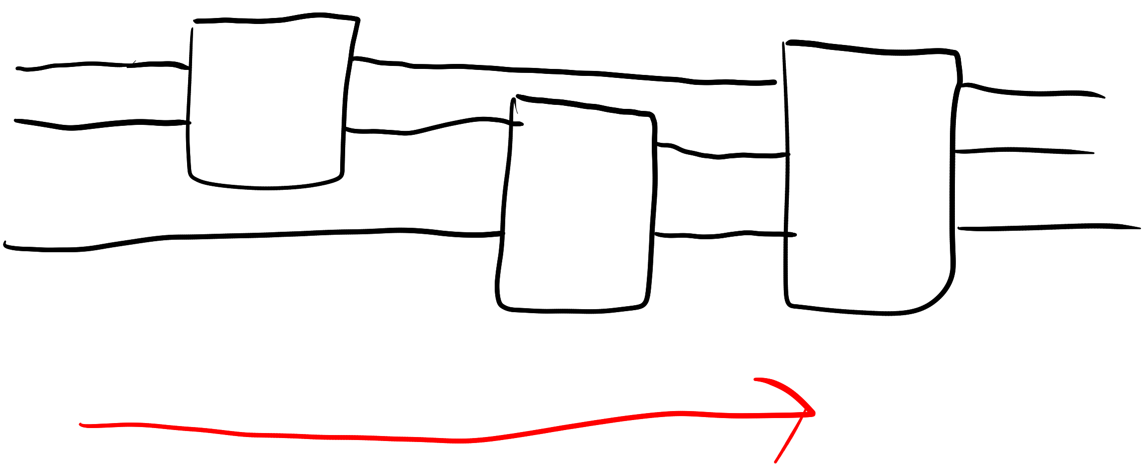
# CMPT 476 Lecture 2

## ... The circuit model...

Last class we learned that quantum computation is linear (i.e. matrices & vectors). Today we'll build a linear algebraic model of classical computing which will extend nicely to probabilistic and then quantum computing.

### (The circuit model)

Circuit models are simple (but powerful) models of computation based around composition of a set of basic operations called gates. Wires are used to connect inputs & outputs of gates. We draw circuits graphically as below



time

(Classical circuits)

In the classical circuit model...

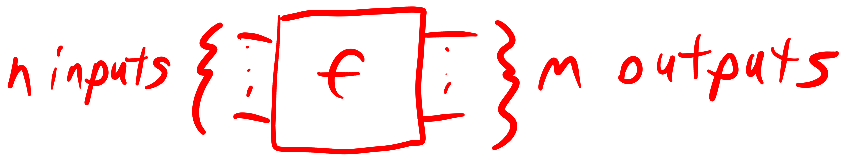
- The state of a bit/wire is 0 or 1
- The state of $n$ bits is a bitstring

$$x \in \{0,1\}^n$$

- Computations are functions

$$f : \{0,1\}^n \longrightarrow \{0,1\}^m$$

As a gate,




Ex.

The NOT gate is a 1-bit function which computes the Boolean not ($\neg$): $\mathrm{NOT}(x) = \neg x$.

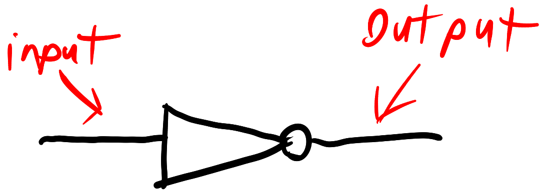
We can write the function explicitly via a truth table.

input

| $x$ | $\mathrm{NOT}(x)$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

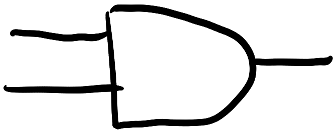
output

We draw a NOT gate as

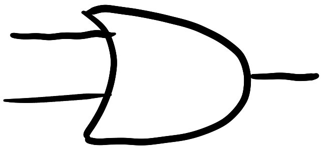


## Ex.

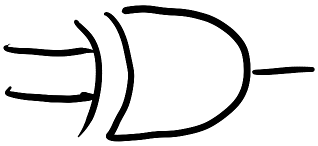
Other common gates are:



$AND(x,y) = x \wedge y$



$OR(x,y) = x \vee y$



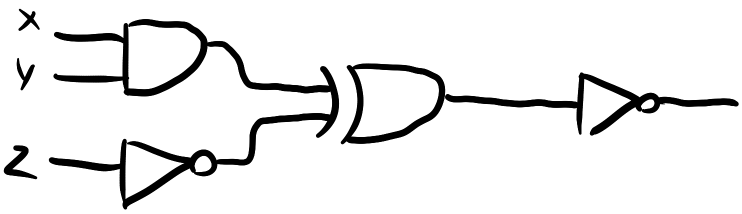
$XOR(x,y) = x \oplus y$

Their truth tables are

| x | y | AND(x,y) | OR(x,y) | XOR(x,y) |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 |

## Ex.

What function does this circuit implement?




We list out each intermediate value in the truth table:

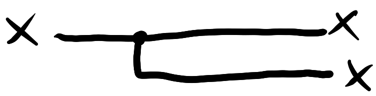
| x | y | z | a=AND(x,y) | b=NOT(z) | c=XOR(a,b) | NOT(c) |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 0 | 1 | 0 |

(Universality)

A set of gates $\Gamma$ is universal for classical computation if for any $n, m > 0$ and $f: \{0,1\}^n \to \{0,1\}^m$, a circuit computing $f$ can be constructed using only gates in $\Gamma$.

(FANOUT)

In classical computing we often assume we can use a bit any number of times in a computation. Formally, this is achieved through the FANOUT or copy gate



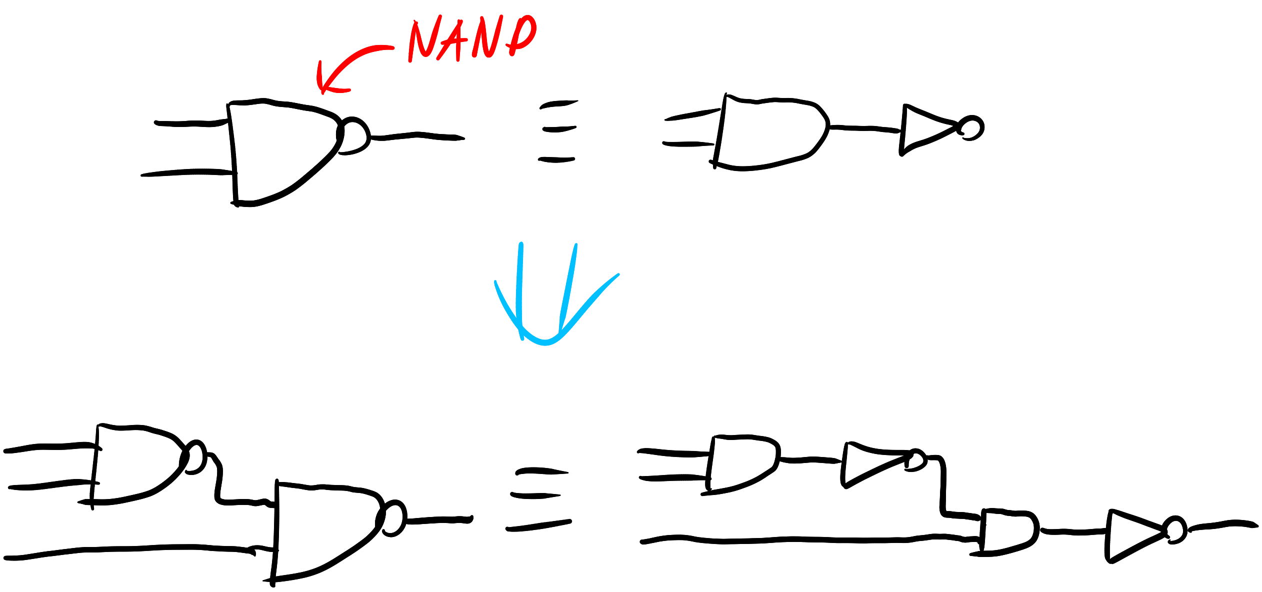

**Thm.**

The set $\{$AND, XOR, NOT, FANOUT$\}$ is universal.

(Translating between gate sets)

We can translate circuits written in one gate set to another by replacing each gate with an equivalent circuit.

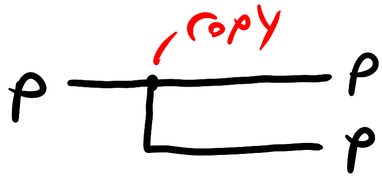
**E.x.**



## (Probabilistic circuits)

What if we wanted to model probabilistic computation? (hence quantum)

We could say a bit with probability $p \in [0,1]$ of being "1" has state $p$. Does this work?




The final state above has probabilities

$$00 \rightarrow (1-p)(1-p)$$
$$01 \rightarrow (1-p)\,p$$
$$10 \rightarrow p\,(1-p)$$
$$11 \rightarrow p^2$$

BUT the states 01 and 10 are impossible!

The problem here is we can't express joint prob distributions. For this we need more degrees of freedom in our state description!

## (Linear algebraic circuits)

In the linear algebraic view, we can represent a bit with probability $p$ in the 1 state as

$$\begin{bmatrix} 1-p \\ p \end{bmatrix}$$

If $p=0$, then we have state $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, or $0$, and if $p=1$, then we have state $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or $1$.

Equivalently, we can describe the state as

$$(1-p)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + p\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(Linear algebraic gates)

Suppose we apply NOT to the probabilistic state $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$. Then we have probability $p_1$ of being 1 and $p_2$ of being 0, or $\begin{bmatrix} p_2 \\ p_1 \end{bmatrix}$. This transformation can be described as a transition matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} p_2 \\ p_1 \end{bmatrix}$$

Note also that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So NOT $\equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ sends 0 to 1 and vice versa

(Multiple bits)

Given two bits $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ and $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ we can write their joint probability distribution as

$$\begin{bmatrix} p_1 q_1 \\ p_1 q_2 \\ p_2 q_1 \\ p_2 q_2 \end{bmatrix} \begin{matrix} \leftarrow 00 \\ \leftarrow 01 \\ \leftarrow 10 \\ \leftarrow 11 \end{matrix}$$

This is called the tensor product

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \otimes \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} p_1 \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \\ p_2 \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} p_1 q_1 \\ p_1 q_2 \\ p_2 q_1 \\ p_2 q_2 \end{bmatrix}$$

(Correlated distributions)

A distribution $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is correlated if it can't be written as a tensor product $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \otimes \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$. Otherwise we say it is separable.

## Ex.

The CNOT or controlled-NOT gate takes 2 bits and applies NOT to the second if and only if the first is 1. As a matrix,

$$\begin{array}{c|c} & \text{input} \rightarrow \; 00 \;\; 01 \;\; 10 \;\; 11 \\ & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix} \leftarrow \text{output} \end{array}$$

Applying CNOT to the state $\begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} \otimes \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{0 state}}$ gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0 \\ 0.75 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0 \\ 0 \\ 0.75 \end{bmatrix} \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix}$$

Now suppose

$$\begin{bmatrix} 0.25 \\ 0 \\ 0 \\ 0.75 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$$

Then $ac = 0.25$, $ad = 0 \Rightarrow d = 0$, but then $bd = 0$, a contradiction, so the distribution is correlated.

(A note on vectors & matrices)

We used the term distribution informally. Formally, a vector (i.e. state) $p \in \mathbb{R}^n$ (real vector space of dim $n$) is a distribution on $\{0, \ldots, n-1\}$ or simply a probability vector if

1. $p_i \geq 0$ for all $i$
2. $\sum_i p_i = 1$ ← Note: this is the 1-norm

If states are probability vectors, then gates should map distributions to distributions. These are exactly the stochastic matrices $A$, which have as columns $A_i$ probability vectors.

Ex.

$$A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

is a stochastic matrix modeling a coin flip. Calling $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ heads and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ tails, we have

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A\begin{bmatrix} p \\ q \end{bmatrix} = \frac{p+q}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{p+q}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

the input state is irrelevant!

In quantum computing, our allowable operations will take on a very similar restriction :)