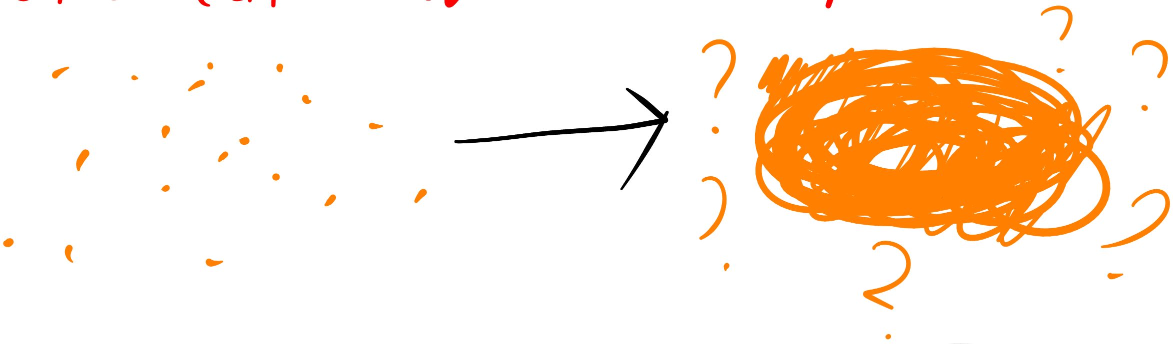
CMPT 476 Lecture 3

From classical to quantum computation



Last class we saw two ^circuit^ models of comp.

- Classical → states: $\{0,1\}^n$
  gates: $\{0,1\}^n \to \{0,1\}^m$
- probabilistic → states: prob. vectors $p \in \mathbb{R}^n$
  gates: stoch. matrices $A \in \mathbb{R}^{n \times n}$

Like these models, quantum computing is built on a notion of **states** and **gates**, with one additional ingredient of **measurements**. As a preview,

- quantum → states: unit vectors $v \in \mathbb{C}^n$
  gates: unitary matrices $U \in \mathbb{C}^{n \times n}$
  measurement: ???

Today we begin to build a model of **quantum computation**, learning about **Dirac notation** and reviewing linear algebra along the way.

(State of a physical system)

The state of an isolated physical system is described by a unit vector in a Hilbert space $\mathcal{H}$.

In the finite-dimensional case (all we care about) we can take $\mathcal{H}$ as $\mathbb{C}^d$ (complex vector space of dimension d).

Let's unpack this postulate!

(Dirac notation)

Let V be a vector space (VS). We write

$$|v\rangle \quad (\text{Ket})$$

to denote a vector in V. The v here is a label and can be anything, e.g. $|\psi\rangle$, $|\square\rangle$, $|☺\rangle$

(Norm)

Let $|v\rangle = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{C}^n$. The (Euclidean) norm of $|v\rangle$ is

$$\| |v\rangle \| = \sqrt{\sum_{i=1}^{n} |a_i|^2}$$

Complex conjugate

Recall that for $a = a_1 + i a_2 \in \mathbb{C}$, $a^* = a_1 - i a_2$

$$|a|^2 = a a^* = a_1^2 + a_2^2$$

(Unit vector)

A unit vector $|v\rangle$ has norm 1 (i.e. $\| |v\rangle \| = 1$)

Ex.

The following are unit vectors:

$$\begin{bmatrix}1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix}\sqrt{3}\\1\end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}$$

the following are not

$$\begin{bmatrix}1\\1\end{bmatrix}, \quad \begin{bmatrix}4\\0\end{bmatrix}, \quad \begin{bmatrix}1-i\\0\end{bmatrix}, \quad \frac{1}{3}\begin{bmatrix}1\\1\\1\\0\end{bmatrix}$$

(Inner products)

Recall that the inner product of

$v = \begin{bmatrix}v_1\\ \vdots\\ v_n\end{bmatrix}$, $u = \begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix} \in \mathbb{C}^n$ is defined as

$$\langle v, u\rangle = \sum_{i=1}^{n} v_i^* u_i = [v_1^* \cdots v_n^*]\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}$$

The row-vector $[v_1^* \cdots v_n^*]$ is the conjugate-transpose or Hermitian conjugate of v, denoted $v^\dagger$ (v-dagger)

We have special notation for $v^\dagger$ in Dirac notation:

$$(|v\rangle)^\dagger = \langle v| \quad (\text{Bra})$$

The we write

$$\langle v, u\rangle = \langle v| \cdot |u\rangle = \langle v|u\rangle \quad (\text{Bra-ket})$$

(Properties of the inner product)

Let $|v\rangle, |u\rangle, |w\rangle \in \mathbb{C}^n$ and $\alpha, \beta \in \mathbb{C}$. Then

1. $\langle v|(\alpha|u\rangle + \beta|w\rangle) = \alpha\langle v|u\rangle + \beta\langle v|w\rangle$
2. $\langle v|v\rangle = \| |v\rangle \|^2 \geq 0$
3. $\langle v|u\rangle = \langle u|v\rangle^*$

useful for computation

(Orthonormal basis)

Let $\mathcal{H}$ be a Hilbert space of dim. n (i.e. $\mathbb{C}^n$)

An orthonormal basis of $\mathcal{H}$ is a set

$$\{|e_i\rangle\} \subseteq \mathcal{H}$$

of size n such that

$$\langle e_i|e_j\rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

Then every vector $|v\rangle \in \mathcal{H}$ can be written as

$$|v\rangle = \sum_{i=1}^{n} a_i |e_i\rangle, \quad a_i \in \mathbb{C}$$

(linear combination)

(Aside: dual spaces)

The Bra $\langle v|$ is really an element of the dual space of $\mathcal{H}$, $\mathcal{H}^*$. The dual space is a VS of linear operators $\langle v|: V \longrightarrow \mathbb{C}$

$$|u\rangle \longmapsto \langle v|u\rangle$$

If $\mathcal{H}$ has orthonormal basis $\{|e_i\rangle\}$, then $\mathcal{H}^*$ has orthonormal basis $\{\langle e_i|\}$

(Qubits (finally (☺)))

The smallest non-trivial Hilbert space is $\mathbb{C}^2$.
We say a qubit has state space $\mathbb{C}^2$

We define the computational basis of $\mathbb{C}^2$ as

$$\left\{ |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

We say that a state

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

is in a superposition of $|0\rangle$ and $|1\rangle$, with amplitudes a & b, respectively.

Ex.

Let $|\psi\rangle = \sqrt{2/3}\,|0\rangle + \frac{i}{\sqrt{3}}|1\rangle$, $|\varphi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$.

1. $$\langle 0|\psi\rangle = \langle 0|\left(\sqrt{2/3}\,|0\rangle + \tfrac{i}{\sqrt{3}}|1\rangle\right)$$
$$= \sqrt{2/3}\,\underbrace{\langle 0|0\rangle}_{1} + \tfrac{i}{\sqrt{3}}\underbrace{\langle 0|1\rangle}_{0}$$
$$= \sqrt{2/3}$$

2. $$\langle \psi|\varphi\rangle = \left(\sqrt{2/3}\,\langle 0| - \tfrac{i}{\sqrt{3}}\langle 1|\right)\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)$$
$$= \sqrt{\frac{2}{3\cdot 2}}\langle 0|0\rangle + \underbrace{\sqrt{\frac{2}{3\cdot 2}}\langle 0|1\rangle - \frac{i}{\sqrt{3\cdot 2}}\langle 1|0\rangle}_{0} - \frac{i}{\sqrt{3\cdot 2}}\langle 1|1\rangle$$
$$= \frac{1}{\sqrt{3}} - \frac{i}{\sqrt{3\cdot 2}}$$

Much easier than explicitly writing vectors when they get large...

(More about qubits)

In principle, any physical system with 2 distinct states $|0\rangle$ and $|1\rangle$ is a qubit. However, to use it as a qubit, we need to maintain coherence of superpositions $a|0\rangle + b|1\rangle$. Examples of such systems include

- Photons in one of 2 different locations/paths
- Photons with horizontal or vertical polarization
- spin-$\frac{1}{2}$ particles (no idea what these are)
- An electron in its lowest energy orbital (its ground state) or a higher energy orbital
- And many more...

(Qudits)

If a system has 3 distinct states, we call it a qutrit and model its state in $\mathbb{C}^3$ with basis

$$\left\{ |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, |2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

More generally we can have systems with d states, called a qudit, which has state space $\mathbb{C}^d$ and basis

$$\left\{ |0\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, |d-1\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

(Aside:

A system of "n" qubits has 2^n states, so we can describe its state space as $\mathbb{C}^{2^n}$ and for $0 \leq i \leq 2^n - 1$ associate $|i\rangle$ with the state

$$|i_n\rangle |i_{n-1}\rangle \cdots |i_1\rangle$$

Where $i_n \cdots i_0$ is the binary expansion of i)

(Operations on states)

What can we do with a quantum state $|\psi\rangle$?

We have two options:

1. Unitary (norm-preserving) linear operators
2. Measurement

We'll leave 1. for now and just talk about 2.

(Measurement)

Given a qubit in the state $\alpha|0\rangle + \beta|1\rangle$, measuring the state produces a result and a new state.

- With probability $|\alpha|^2$, result is 0
  new state is $|0\rangle$
- With probability $|\beta|^2$, result is 1
  new state is $|1\rangle$

Intuition is measurement collapses the uncertain state $\alpha|0\rangle + \beta|1\rangle$ to a particular state $|0\rangle$ or $|1\rangle$.

Ex.

Let $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$.

Measuring $|\psi\rangle$ produces:

$|0\rangle$ with probability $\left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4}$

$|1\rangle$ with probability $\left|\frac{i}{2}\right|^2 = \frac{i}{2} \cdot \frac{-i}{2} = \frac{1}{4}$

We can verify that $\frac{3}{4} + \frac{1}{4} = 1$:)

What would it mean to measure a qutrit?

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$$

Same thing!

- Get result 0 and state $|0\rangle$ w/ prob. $|\alpha|^2$
- Result 1 and state $|1\rangle$ w/ prob. $|\beta|^2$
- Result 2 and state $|2\rangle$ w/ prob. $|\gamma|^2$

In principle we can measure a state $|\psi\rangle \in \mathcal{H}$ over any orthonormal basis of $\mathcal{H}$. We will see why we can do so in practice later on.

(Measurement over a basis)

More generally, given a basis $\{|e_i\rangle\}$ of $\mathbb{C}^d$, measuring the state $\sum_i a_i|e_i\rangle$ produces the result i and state $|e_i\rangle$ with probability $|a_i|^2$

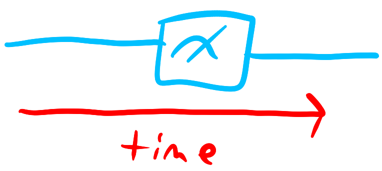
Observe that if $|\psi\rangle = \sum_i a_i|e_i\rangle$, then

$$a_i = \langle e_i|\psi\rangle$$

and hence $|a_i|^2 = |\langle e_i|\psi\rangle|^2$

Notation

In circuit diagrams, we denote a qubit/qudit by a line/wire, and measurement in the computational basis by $\boxed{\nearrow}$, e.g.



Ex.

Another common basis of $\mathbb{C}^2$ is the hadamard basis

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Observe that

$$\langle + | + \rangle = \frac{1}{2}(\langle 0|0\rangle + \langle 0|1\rangle + \langle 1|0\rangle + \langle 1|1\rangle) = 1$$

$$\langle - | - \rangle = \frac{1}{2}(\langle 0|0\rangle - \langle 0|1\rangle - \langle 1|0\rangle + \langle 1|1\rangle) = 1$$

$$\langle + | - \rangle = \frac{1}{2}(\langle 0|0\rangle - \langle 0|1\rangle + \langle 1|0\rangle - \langle 1|1\rangle) = 0$$

and hence $\{|+\rangle, |-\rangle\}$ is orthonormal. Also note that

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

$$|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

So measuring the state $|0\rangle$ in the $\{|+\rangle, |-\rangle\}$ basis produces

- $|+\rangle$ with prob. $1/2$
- $|-\rangle$ with prob. $1/2$

Geometrically, the $|+\rangle, |-\rangle$ basis is a 45° rotation of $|0\rangle, |1\rangle$