CMPT 476 Lecture 5

Working with a qubit

We know now that an isolated quantum system corresponds to a d-dimensional Hilbert space $\mathbb{C}^d$ and we can affect its state by applying either:

- Unitary operations 
- Measurements 

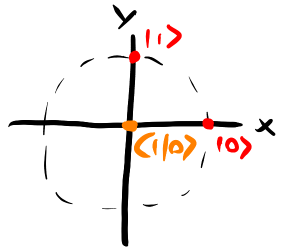
Before we move on to multiple qubits, let's see what kind of quantum effects we can witness with a single qubit.

(Quantum Zeno effect)

Measurement can effect states in a strange way. For instance, given the state $|\psi\rangle = |0\rangle$, what is the probability of measuring $|1\rangle$?

$$|\langle 1|\psi\rangle|^2 = 0$$

Geometrically, this is the projection onto the y axis



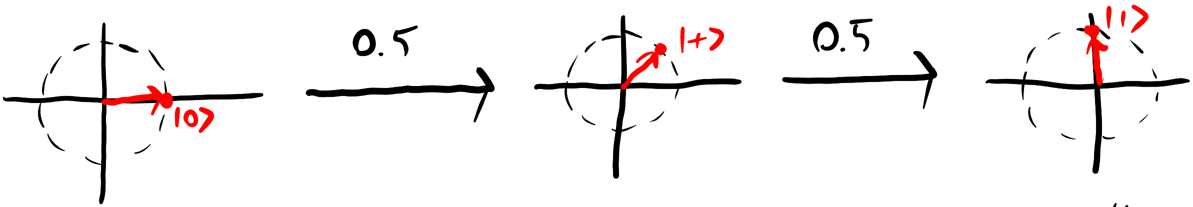
However, suppose we measured first in the $|+\rangle, |-\rangle$? we would get state $|+\rangle$ with probability

$$|\langle +|\psi\rangle|^2 = \left|\tfrac{1}{\sqrt{2}}(\langle 0| + \langle 1|)|0\rangle\right|^2$$
$$= \left|\tfrac{1}{\sqrt{2}}(\langle 0|0\rangle + \langle 1|0\rangle)\right|^2$$
$$= \tfrac{1}{2}$$

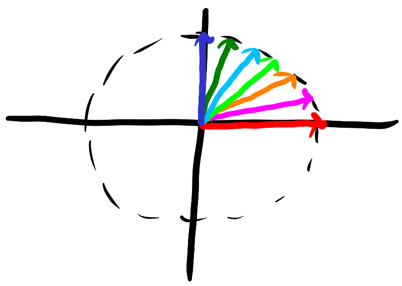
Now when we measure what is the prob. of getting $|1\rangle$?

$$|\langle 1|\psi\rangle|^2 = \left|\tfrac{1}{\sqrt{2}}\right|^2 = \tfrac{1}{2}$$

In effect we can change a state by measuring in different bases



If we make the angle between bases small enough, we can do this with high probability

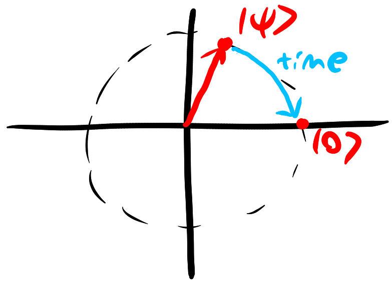


(A watched pot never boils)

A similar effect can be used to control decoherence. Suppose we have a superposition

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

which decoheres over time to the $|0\rangle$ state, i.e.



then we can keep it in state $|\psi\rangle$ with high probability if we repeatedly measure in the basis

$$\{|\psi\rangle, |\psi^\perp\rangle = b^*|0\rangle - a^*|1\rangle\}$$

Of course, this assumes we know $|\psi\rangle$ ahead of time, so it's not really an uncertain state in a useful sense. We will see that quantum algorithms require genuine superpositions where a and b are not known a priori to achieve speed ups.

(Elitzur-Vaidman Bomb)

A closely related thought experiment is the Elitzur-Vaidman Bomb. Here are the rules:

A suspicious man hands you and a friend two boxes. With them is a note that reads

> "Do you want to play a game? In one box is a bomb triggered by a horizontally polarized photon. If you open the box it will explode, and if you do nothing I will trigger the bomb. Find out which box has the bomb or many will die..."

So, the rules are:

1. We can send a photon in state $|\psi\rangle$ into the box
2. If there is no bomb we get state $|\psi\rangle$ back
3. If there is a bomb, $|\psi\rangle$ is measured
   - 3.1 If the result is $|0\rangle$ vertical the bomb doesn't go off
   - 3.2 If $|1\rangle$ horizontal the bomb goes off

The thought experiment shows that with high probability a quantum system can win the game.

Suppose our photon decoheres towards $|1\rangle$ at an angle of ε each second.

If we start in state $|0\rangle$, and send it into the box every second, after $\frac{\pi}{2\varepsilon}$ seconds,

1. If no bomb, state has rotated $\frac{\pi}{2\varepsilon}\cdot\varepsilon = 90^\circ$ to $|1\rangle$
2. If there is a bomb, each time the measurement snaps us back to the $|0\rangle$ state if it doesn't trigger, which happens with probability

$$\sin^2\varepsilon \approx \varepsilon^2 \text{ for small } \varepsilon$$

(remember: $e^{i\varepsilon} = \cos\varepsilon + i\sin\varepsilon$)

So we only set off the bomb with probability

$$\frac{\pi}{2\varepsilon} \cdot \varepsilon^2 = \frac{\pi}{2}\varepsilon$$

and if not, we end with state $|0\rangle$ in case of a bomb, and $|1\rangle$ if no bomb. Pretty cool!

(Distinguishing states)

Now that we've had our fun, a more practical question is: given a qubit $|\psi\rangle$, can we determine what $|\psi\rangle$ is? It should be fairly obvious that we can't in general if $|\psi\rangle$ is unknown, because measuring will collapse the state. In analogy to probabilistic computation, we can't determine the probability distribution of a bit (i.e. its state) by observing its value (i.e. measuring). However, in some cases we can determine with high probability which of two possible states we have.

Ex.

Suppose you're handed a qubit $|\psi\rangle$ and told that either $|\psi\rangle = |0\rangle$ or $|\psi\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Can you determine which case it is?

A simple protocol with a one-sided error is to measure in the computational basis and guess

- $|0\rangle$ if the result is 0
- $|+\rangle$ if the result is 1

If $|\psi\rangle = |0\rangle$, then this protocol always guesses correctly. If $|\psi\rangle = |+\rangle$, then we measure 1 with 50% probability and hence guess correctly with 50% probability.

What if the person who gave you $|\psi\rangle$ is trying to trick you and intentionally gives $|+\rangle$ in anticipation of this strategy? In this case it's better to have a two-sided error. We can do this by making our guess probabilistically to account for the imbalance.

Here's a two-sided error protocol:

- If result is 0, guess $|0\rangle$ with probability $2/3$ and $|+\rangle$ with probability $1/3$
- If result is 1, guess $|+\rangle$ as before

Now if $|\psi\rangle = |0\rangle$, we guess correctly with prob. $2/3$ and if $|\psi\rangle = |+\rangle$, then we guess correctly with prob.

$$\underbrace{\frac{1}{2}\cdot\frac{1}{3}}_{\text{measure } 0} + \underbrace{\frac{1}{2}}_{\text{measure } 1} = 2/3$$

(Global vs relative phase)

One broad class of states which cannot be distinguished are those which differ by a global phase $e^{i\theta}$

Ex.

If $|\psi\rangle$ is a state, then so is $|\psi'\rangle = e^{i\theta}|\psi\rangle$ for any θ:

$$\langle\psi'|\psi'\rangle = (e^{-i\theta}\langle\psi|)(e^{i\theta}|\psi\rangle) = \langle\psi|\psi\rangle$$

$|\psi\rangle$ and $|\psi'\rangle$ are said to be related by a global phase, and are indistinguishable by measurement:

$$\begin{aligned}|\langle e_i|\psi'\rangle|^2 &= |e^{i\theta}\langle e_i|\psi\rangle|^2 \\ &= (e^{i\theta}\langle e_i|\psi\rangle)(e^{-i\theta}\langle e_i|\psi\rangle) \\ &= |\langle e_i|\psi\rangle|^2\end{aligned}$$

On the other hand, a relative phase, e.g.

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |\psi'\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

relative phase on $|1\rangle$ state

is distinguishable!

$$|\langle +|\psi\rangle|^2 = 1 \qquad |\langle +|\psi'\rangle|^2 = 0$$

$$|\langle -|\psi\rangle|^2 = 0 \qquad |\langle -|\psi'\rangle|^2 = 1$$

So given either $|\psi\rangle$ or $|\psi'\rangle$, you can determine which you have by measuring in the $\{|+\rangle, |-\rangle\}$ basis.

(The bloch sphere)

A final note about qubits is that we can visualize their states as lying on a 3-dimensional unit sphere called the Bloch sphere.

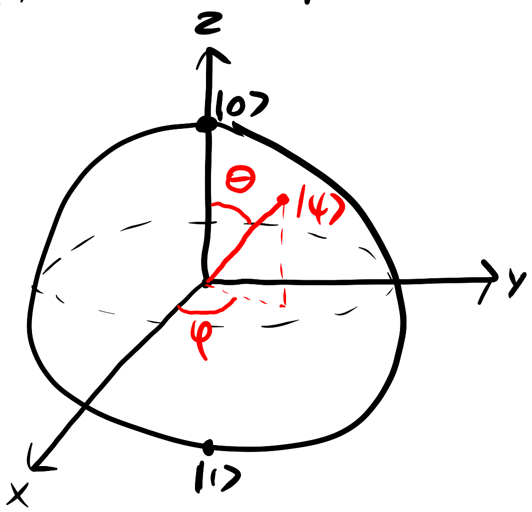
Since the state of a single qubit is

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$$

we may write it up to global phase as

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle$$

These angles (θ, φ) define a point on the unit 3-sphere like so



In this picture, we can view the relative phase φ as rotating our state around the z-axis. While the Bloch sphere has limited use in higher-dimensional or multi-qubit systems, it can be useful for understanding single-qubit unitaries, which as we will see later on correspond to rotations of the Bloch sphere.