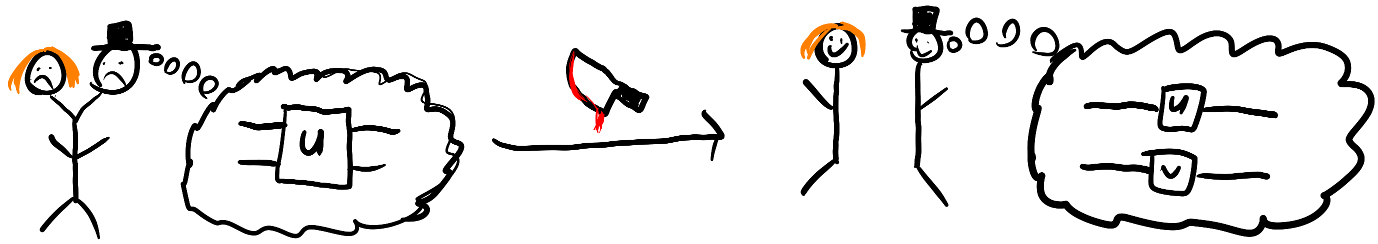
CMPT 476 Lecture 6

A few qubits more



What do we know about QM so far?

- States $|\psi\rangle = \sum_i a_i |i\rangle = \begin{bmatrix} a_0 \\ \vdots \\ a_{d-1} \end{bmatrix} \in \mathbb{C}^d$ s.t. $\sum |a_i|^2 = 1$

- Measurement sends $|\psi\rangle$ to $|i\rangle$ with prob. $|a_i|^2$
- Gates send $|\psi\rangle \longrightarrow U|\psi\rangle$ where $U \in \mathcal{L}(\mathbb{C}^d)$ is unitary ($U^\dagger = U^{-1} \iff U^\dagger U = UU^\dagger = I$) and has matrix $\sum_{ij} u_{ij} |i\rangle\langle j|$, or

$$\begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0d-1} \\ u_{10} & u_{11} & & \\ \vdots & & \ddots & \\ u_{d-10} & & & u_{d-1d-1} \end{bmatrix}$$

An example of a unitary on $\mathbb{C}^4$ is the CNOT gate from lecture 2

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Today we will learn how to build composite systems like $\mathbb{C}^4$ from two qubits (i.e. $\mathbb{C}^2$), how to operate on them, and some implications.

How does a 4-dimensional system arise? Could be

- An electron in one of 4 orbitals (is this even possible?)
- A photon with a polarization ($\updownarrow$ or $\leftrightarrow$) and path ($\uparrow$ or $\rightarrow$)
- Two physically separate qubits

Each of the above has 4 physical states, but drastically different dynamics. The last one in particular can only evolve according to local operations on either qubit. We shall see that with entanglement, local operations have surprising power.

First, suppose Alice's qubit has state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and Bob's has state $|\varphi\rangle = \gamma|0\rangle + \delta|1\rangle$. If both Alice and Bob measure their qubits, what is the distribution of outcomes?

- $|0\rangle$ and $|0\rangle \longrightarrow |\alpha|^2 \cdot |\gamma|^2$
- $|0\rangle$ and $|1\rangle \longrightarrow |\alpha|^2 \cdot |\delta|^2$
- $|1\rangle$ and $|0\rangle \longrightarrow |\beta|^2 \cdot |\gamma|^2$
- $|1\rangle$ and $|1\rangle \longrightarrow |\beta|^2 \cdot |\delta|^2$

We saw in lecture 2 that this is the joint prob. distribution

$$\underbrace{\begin{bmatrix} |\alpha|^2 \\ |\beta|^2 \end{bmatrix} \otimes \begin{bmatrix} |\gamma|^2 \\ |\delta|^2 \end{bmatrix}}_{\text{tensor product}} = \begin{bmatrix} |\alpha|^2|\gamma|^2 \\ |\alpha|^2|\delta|^2 \\ |\beta|^2|\gamma|^2 \\ |\beta|^2|\delta|^2 \end{bmatrix}$$

Thinking of quantum amplitudes as "probabilities" we can wonder if

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \otimes \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{bmatrix}$$

is a sensible representation of the 4-dimensional state. Indeed,

$$|\alpha\beta|^2 = |\alpha|^2|\beta|^2 \Rightarrow |\alpha\gamma|^2 + |\alpha\delta|^2 + |\beta\gamma|^2 + |\beta\delta|^2$$
$$= |\alpha|^2(|\gamma|^2 + |\delta|^2) + |\beta|^2(|\gamma|^2 + |\delta|^2)$$
$$= 1$$

(State of a composite system, pt 1)

Given two systems in states $|\psi\rangle$ and $|\varphi\rangle$, their joint state is $|\psi\rangle \otimes |\varphi\rangle$

Going back to Alice and Bob, what if they brought their qubits together (i.e. $|\psi\rangle \otimes |\varphi\rangle$) and then applied a CNOT to the joint state?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{bmatrix} = \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\delta \\ \beta\gamma \end{bmatrix}$$

Suppose $\alpha = \frac{1}{\sqrt{2}}$, $\beta = \frac{1}{\sqrt{2}}$, $\gamma = 1$, $\delta = 0$. Then

$$\begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\delta \\ \beta\gamma \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \neq \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$$

(Why?)

Recall that in a probabilistic setting we said that Alice & Bob's states are correlated if the joint state $|\psi\rangle \neq \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix}$. In Quantum mechanics we say their states are entangled. Since the above postulate says nothing about composite systems with entangled states, we need to generalize it a bit using more linear algebra.

(Tensor products)

Let V, W be two vector spaces (e.g. Hilbert spaces). The **tensor product** $V \otimes W$ is a vector space such that:

1. $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$

$\otimes$ is bilinear:

2. $v \otimes w \in V \otimes W \quad \forall v \in V, w \in W$
3. $(v+u) \otimes w = v \otimes w + u \otimes w$
4. $v \otimes (u+w) = v \otimes u + v \otimes w$
5. $(cv) \otimes w = v \otimes (cw) = c(v \otimes w)$, c scalar

The definition above describes tensor products abstractly (and is in fact missing a crucial abstract property). Since we work in finite dimensions it's easier to simply give a basis for $V \otimes W$.

(Basis of $V \otimes W$)

Let V & W be V.S. with orthonormal bases $\{|e_i\rangle\}$, $\{|f_j\rangle\}$ respectively. Then $V \otimes W$ has an orthonormal basis

$$\{|e_i\rangle \otimes |f_j\rangle = |e_i\rangle|f_j\rangle\}$$

Ex.

$\mathbb{C}^2 \otimes \mathbb{C}^2$ has basis $\{|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle\}$

If $|\psi\rangle = i|1\rangle$ and $|\varphi\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle$, then

$$|\psi\rangle \otimes |\varphi\rangle = i|1\rangle \otimes \left(\frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle\right)$$
$$= \frac{i}{\sqrt{5}}|1\rangle|0\rangle + \frac{2i}{\sqrt{5}}|1\rangle|1\rangle$$

(Multi-qubit quantum computing & notation)

$\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^4$ look very similar...

	$\mathbb{C}^2 \otimes \mathbb{C}^2$	$\mathbb{C}^4$
dim	$2 \times 2 = 4$	4
basis	$\{\|0\rangle\|0\rangle, \|0\rangle\|1\rangle, \|1\rangle\|0\rangle, \|1\rangle\|1\rangle\}$	$\{\|0\rangle, \|1\rangle, \|2\rangle, \|3\rangle\}$ $\Downarrow$ Binary expansion $\{\|00\rangle, \|01\rangle, \|10\rangle, \|11\rangle\}$

We say $\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$ via the isomorphism $|i\rangle|j\rangle \longleftrightarrow |ij\rangle$. In practice we view $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$ and write the following equivalenty:

$$e_i = |i\rangle = |\underbrace{i_1 i_0}_{\text{binary exp.}}\rangle = |i_1\rangle|i_0\rangle = |i_1\rangle \otimes |i_0\rangle$$

More generally, for n qubits,

$$\overbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}^{n \text{ times}} \simeq \mathbb{C}^{2^n}$$

$$|i\rangle = |i_{n-1} \cdots i_0\rangle = |i_{n-1}\rangle \cdots |i_0\rangle = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th entry}$$

And for n qudits of dimension d,

$$\mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d \simeq \mathbb{C}^{d^n}$$

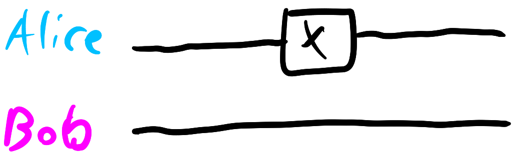
(More notation)

To make it even more confusing, depending on the context the following are the same

$$|\psi\rangle \otimes |\varphi\rangle, \quad |\psi\rangle|\varphi\rangle, \quad |\psi, \varphi\rangle$$

(local operations)

Back to our example, suppose Alice wants to apply a NOT/X gate to her qubit. Diagrammatically,



We haven't used circuit diagrams much yet because we only had 1 qubit. They'll factor in more now. Note that

$$-\boxed{A}-\boxed{B}- = BA$$

since time flows left to right

If their states are not entangled we might view this circuit as sending

$$\underbrace{|\psi\rangle}_{\text{Alice}} \otimes \underbrace{|\varphi\rangle}_{\text{Bob}} \longrightarrow (X|\psi\rangle) \otimes |\varphi\rangle$$

If they are entangled, we need a way of describing "X on Alice's qubit" as a linear operator over $\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$.

(Tensor product of operators)

Let $A: V \to V$, $B: W \to W$ be linear operators on V & W. Then $A \otimes B : V \otimes W \longrightarrow V \otimes W$ is the linear operator defined by

$$(A \otimes B)|v\rangle \otimes |w\rangle = (A|v\rangle) \otimes (B|w\rangle)$$

As a matrix, this corresponds to

Barf! ×́

$$\begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \otimes \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} = \begin{bmatrix} A_{00}\begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} & A_{01}\begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} \\ A_{10}\begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} & A_{11}\begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} \end{bmatrix}$$

Ex.

Recall that $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Compute:

1. $X \otimes I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

2. $I \otimes X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

3. $Z \otimes Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. $(X \otimes I)(|0\rangle \otimes |1\rangle) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |1\rangle \otimes |1\rangle$

(Properties of tensor products)

1. $s(A \otimes B) = sA \otimes B = A \otimes sB$
2. $(A+B) \otimes C = A \otimes C + B \otimes C$ (left distributive)
3. $A \otimes (B+C) = A \otimes B + A \otimes C$ (right distributive)
4. $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ (associative)
5. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$ ($\dagger$-distributivity)
6. $(A \otimes B)(C \otimes D) = AC \otimes BD$ (bi-functor)

Ex.

$$(|1\rangle \otimes |0\rangle)^\dagger (|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

$= (\langle 1| \otimes \langle 0|)(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$ ($\dagger$-dist)

$= ((\langle 1| \otimes \langle 0|)(|0\rangle \otimes |1\rangle) - ((\langle 1| \otimes \langle 0|)(|1\rangle \otimes |0\rangle)$ (dist.)

$= (\langle 1|0\rangle \otimes \langle 0|1\rangle) - (\langle 1|1\rangle \otimes \langle 0|0\rangle)$ (bi-func)

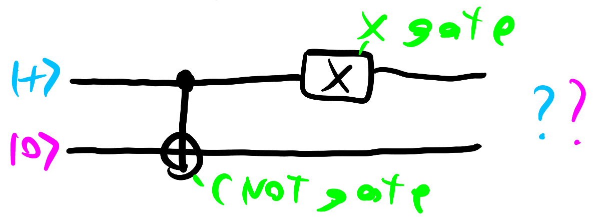
$= -1$ (surprised?)

(Back to our friends)

We now have all the ingredients to compute the final state of our Alice and Bob example. To recap:

1. Alice starts in state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$,
   Bob in state $|0\rangle$
2. Alice and Bob jointly apply CNOT
3. Alice separates her qubit and applies X

We can write this as a Circuit



Recalling that $CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, the final state is

$$\Bigg(\overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{\boxed{X}}\Bigg) \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}^{\oplus} \Bigg(\overbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{|+\rangle = \;,\; |0\rangle =}\Bigg)$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

Alt. calculation

$$CNOT: \begin{aligned} |00\rangle &\longmapsto |00\rangle \\ |01\rangle &\longmapsto |01\rangle \\ |10\rangle &\longmapsto |11\rangle \\ |11\rangle &\longmapsto |10\rangle \end{aligned}$$

$$X: \begin{aligned} |0\rangle &\longmapsto |1\rangle \\ |1\rangle &\longmapsto |0\rangle \end{aligned}$$

SO

$$(X \otimes I)\, CNOT\left(\tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|10\rangle\right)$$

$$= (X \otimes I)\left(\tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle\right)$$

$$= \tfrac{1}{\sqrt{2}}|10\rangle + \tfrac{1}{\sqrt{2}}|01\rangle$$

Dirac is best ☺

(Is quantum computing stochastic?)

Measurement statistics in the Alice and Bob example are identical whether we measure first and proceed stochastically, taking

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \xrightarrow{\text{measure}} \frac{1}{2}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

We might wonder if all QC can be modeled by stochastic matrices. The answer lies in the hadamard gate

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

which has the measurement statistics of a coin flip:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow{\text{measure}} \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

However, unlike a classical coin flip

$$\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

flipping the coin again gets us back to the original state:

$$H\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) = |0\rangle \xrightarrow{\text{measure}} \begin{bmatrix}1\\0\end{bmatrix}$$

In contrast,

$$\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

The reason is negative amplitudes giving rise to interference. We will examine this and other quantum effects in the coming weeks.