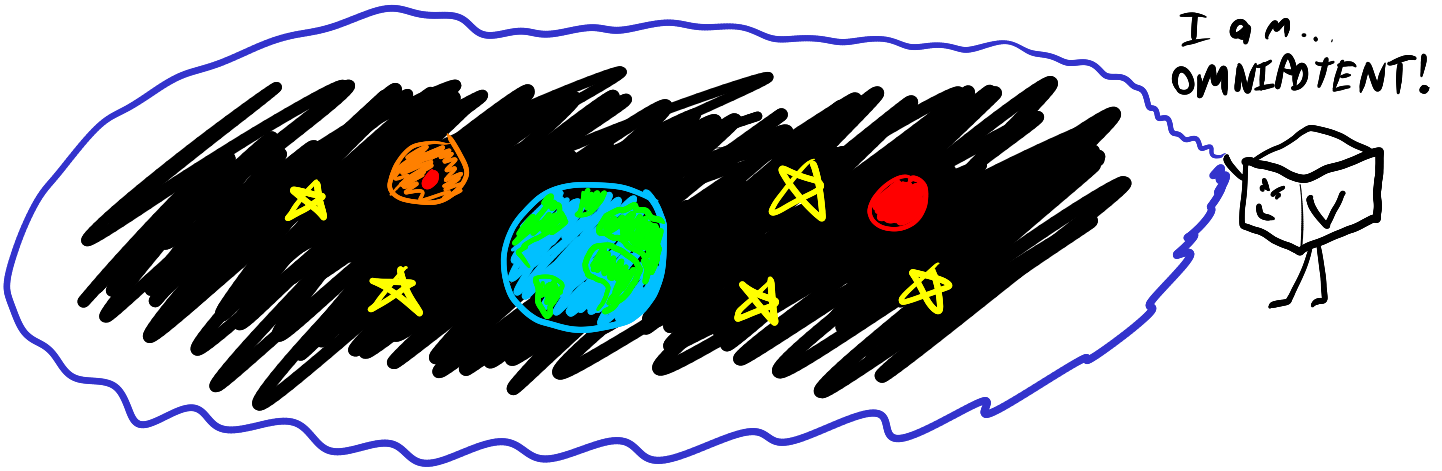
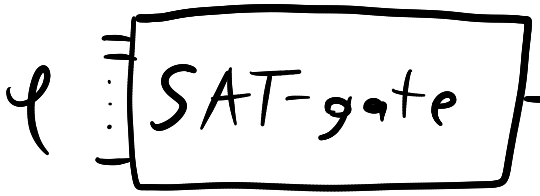
CMPT 476 Lecture 14

Gate sets & quantum universality



In our previous discussion of the quantum circuit model, we made no mention of the **gateset**. However, in the classical circuit model, if our gateset consists of **every classical function**, then all complexity classes would collapse into constant time!



In the quantum circuit model, the situation is even more extreme — allowing **any unitary matrix** as a gate gives **all "Turing degrees" in polynomial time!**

To avoid problems like these, we typically restrict circuit models to a **finite universal gateset**. But is there such a gateset for quantum computing?

Read on to find out...

(A compendium of common gates)

Pauli gates:

identity
$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

bit flip
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

phase flip
$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

bit & phase flip
$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Clifford gates (includes Pauli gates):

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
Hadamard or branch gate

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$
phase gate

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
controlled-NOT or quantum XOR

Gottesman-Knill theorem:

Circuits consisting of only Clifford gates can be efficiently simulated classically

(And in particular are not universal)

Non-Clifford gates:

$$T = \sqrt{S} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

$$Toffoli = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Rotation gates:

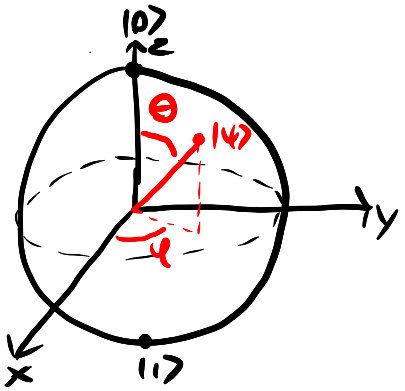
$$R_z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$ z rotation

$$R_x(\theta) = \begin{bmatrix} \cos\theta/2 & i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{bmatrix}$$ x rotation

$$R_y(\theta) = \begin{bmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{bmatrix}$$ y rotation

(Single qubit gates & the Bloch sphere)

Recall that the state of a qubit is a vector on the Bloch sphere



Since a single-qubit unitary maps points on the Bloch sphere to points on the Bloch sphere in a reversible manner, every single-qubit unitary is a rotation of the Bloch sphere.

Ex.

The X gate is a rotation around the x-axis of 180°. It maps $|0\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |0\rangle$.

(Rotation gates)

The rotation gates $R_x(\theta)$, $R_z(\theta)$, $R_y(\theta)$ are rotations of angle θ around the x, z, and y axes respectively. They arise as matrix exponentials of Pauli gates:

$$R_x(\theta) = e^{-i\theta/2 X}$$

$$R_z(\theta) = e^{-i\theta/2 Z}$$

$$R_y(\theta) = e^{-i\theta/2 Y}$$

What does this mean?

(Important aside: operator functions)

Let $f: \mathbb{C} \to \mathbb{C}$ and $A: \mathcal{H} \to \mathcal{H}$ be defined as

$$A = \sum_i a_i |e_i\rangle\langle e_i|$$

for some orthonormal basis $\{|e_i\rangle\}$ of $\mathcal{H}$. We say A is diagonal in the basis $\{|e_i\rangle\}$, and

$$f(A) = \sum_i f(a_i)|e_i\rangle\langle e_i|$$

Ex.

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is diagonal in the computational basis, and specifically

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Then

$$\begin{aligned} \sqrt{Z} &= \sqrt{1}|0\rangle\langle 0| + \sqrt{-1}|1\rangle\langle 1| \\ &= |0\rangle\langle 0| + i|1\rangle\langle 1| \\ &= \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \\ &= S \end{aligned}$$

Likewise,

$$\begin{aligned} R_Z(\theta) = e^{-i\theta/2 Z} &= e^{-i\theta/2}|0\rangle\langle 0| + e^{i\theta/2}|1\rangle\langle 1| \\ &= \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix} \end{aligned}$$

What if A is not diagonal in the computational basis?

(The spectral theorem, for real this time)

An operator $A: \mathcal{H} \to \mathcal{H}$ is **normal** if & only if

$$AA^\dagger = A^\dagger A$$

All unitary or Hermitian operators are normal (why?)

The **Spectral theorem** states that every (finite dimensional) normal operator A can be written as

$$A = P \Lambda P^\dagger$$

Where:

1. Λ is diagonal as a matrix and encodes the **eigenvalues** of A
2. P is unitary and has as columns unit **eigenvectors** of A

we say that A **is diagonalizable** and

$$A = \sum_i \lambda_i P|i\rangle\langle i| P^\dagger$$

the **diagonalization** of A.

Ex.

Recall that X has eigenvectors $|+\rangle, |-\rangle$:

$$X|+\rangle = X\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle = |+\rangle$$
$$X|-\rangle = X\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle = -|-\rangle$$

Since $H = [|+\rangle \; |-\rangle]$, H **diagonalizes** X

$$X = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = HZH$$

(Back to exponentials)

Observe that if $A = P \Lambda P^\dagger$ is normal, then

$$f(A) = \sum_i f(\lambda_i) P|i\rangle\langle i| P^\dagger$$
$$= P \left(\sum_i f(\lambda_i) |i\rangle\langle i| \right) P^\dagger$$
$$= P f(\Lambda) P^\dagger$$

Ex.

Since $X = HZH$, $R_X(\theta) = H e^{-i\theta/2 Z} H = H R_Z(\theta) H$.

A tedious calculation would show that

$$R_X(\theta) = \begin{bmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{bmatrix}$$

(A note on exponentials)

We won't make use of matrix exponentials often, but they are extremely important to Hamiltonian simulation, so it's useful to know and get comfortable with.

(Euler angle decomposition)

Let U be a single-qubit unitary. Then there exist angles $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$U = e^{i\alpha} R_Z(\beta) R_X(\gamma) R_Z(\delta)$$

The above works with any other non-parallel axes as well, like Z & Y. This is a quantum re-statement of the very old fact that any rotation in 3-d space can be implemented as a sequence of 3 rotations in 2 planes.

(Multi-qubit gates)

Since single-qubit gates can't entangle qubits, we know we'll need at least one multi-qubit gate for universal quantum computing.

One possibility is the SWAP gate, which we haven't seen before:

$$= \text{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The SWAP gate swaps two qubits, and hence is not entangling.

$$\text{SWAP}\left(\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \otimes \begin{bmatrix} \gamma \\ \delta \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{bmatrix}$$

$$= \begin{bmatrix} \alpha\gamma \\ \beta\gamma \\ \alpha\delta \\ \beta\delta \end{bmatrix}$$

$$= \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \otimes \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

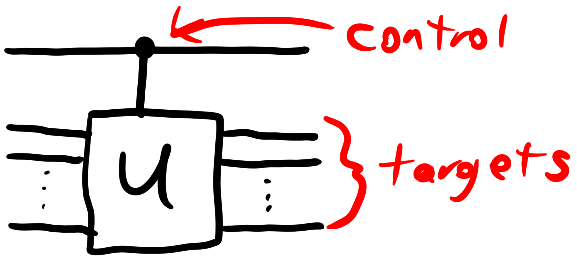
On the other hand, controlled gates usually are entangling. We've seen one such gate already: the CNOT gate.

(Controlled gates)

Let U be an n-qubit unitary. The controlled-U gate c-U is an $n+1$-qubit unitary such that

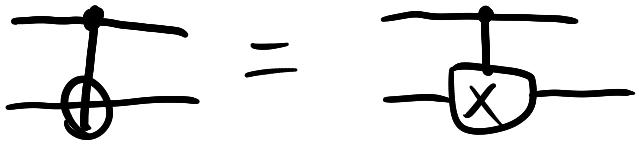
$$\text{c-}U|0\rangle|\psi\rangle = |0\rangle|\psi\rangle$$
$$\text{c-}U|1\rangle|\psi\rangle = |1\rangle(U|\psi\rangle)$$

We draw a controlled-U gate as



Ex.

The CNOT gate is the controlled-NOT, i.e.



It sends $|0\rangle|x\rangle \longmapsto |0\rangle|x\rangle$ and $|1\rangle|x\rangle \longmapsto |1\rangle|1 \oplus x\rangle$, or more concisely

$$\text{CNOT}: |x\rangle|y\rangle \longmapsto |x\rangle|x \oplus y\rangle$$

Ex.

The Toffoli gate is also a controlled gate,

$$\text{Toffoli} = c\text{-CNOT}$$

In particular, noting that

$$\text{Toffoli}\,|x\rangle|y\rangle|z\rangle = |x\rangle|y\rangle|z \oplus xy\rangle$$

we have

$$\text{Toffoli}\,|0\rangle|y\rangle|z\rangle = |0\rangle|y\rangle|z\rangle$$
$$\text{Toffoli}\,|1\rangle|y\rangle|z\rangle = |1\rangle|y\rangle|z \oplus y\rangle$$
$$= |1\rangle(\text{CNOT}|y\rangle|z\rangle)$$

Ex.

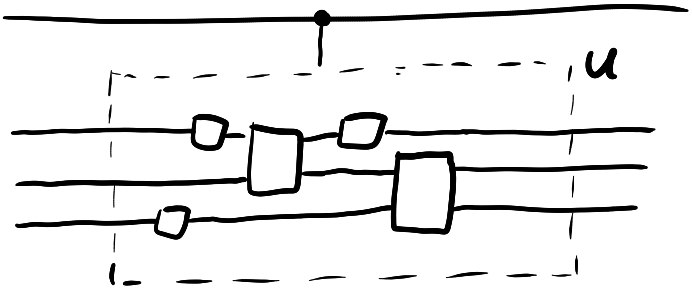
The controlled-Z or CZ gate is sometimes drawn

and has matrix

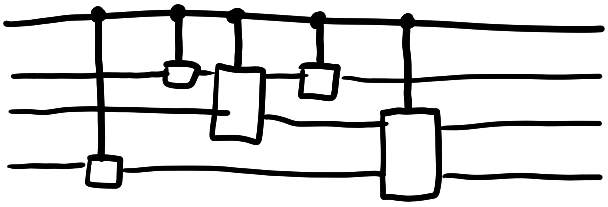
$$CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(Controlled gate implementations)

A common pattern in quantum computation is to take some unitary implemented as a circuit and control it.



We can do this by controlling each gate in U individually



What if we can't decompose U further?

(Theorem, Barenco et. al.)

Every controlled unitary can be implemented as a circuit consisting of CNOT and single-qubit unitaries

(Corollary)

{CNOT} + Single-qubit unitaries is universal for quantum computing.

The proof of the above is not hard, but outside the scope of this course. In fact, CNOT is not unique here and can be replaced with any entangling gate. In other words, Entanglement + local ops is universal

(Approximate universality)

We now have a universal gate set, but it is not very satisfactory since it contains infinitely many gates. For practical, programmable quantum computers, it will be necessary to restrict the gate set to a finite set as in classical computing. Part of this is due to error correction which we will get to late in the course.

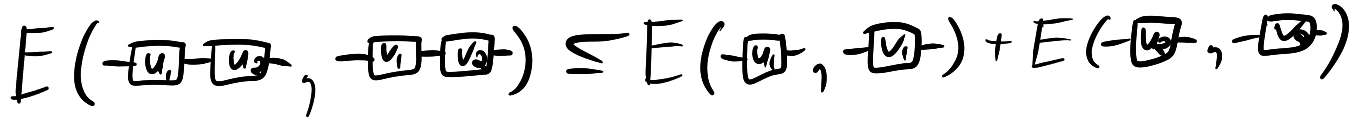
Unlike the classical case, there is no finite gate set which can implement every unitary. The reason is $\mathbb{C}$, and hence 2×2 unitaries, is uncountably infinite. Instead, we may ask whether there exists a finite gate set which can approximate every unitary matrix.

(Gate approximation error)

A unitary U approximates another unitary V with error

$$E(U,V) = \max_{|\psi\rangle} \|(U-V)|\psi\rangle\|$$

An important property which you will prove in homework is that approximation error is additive, in that.



A practical consequence is that if every gate in a circuit is approximated to error $\frac{\varepsilon}{k}$ for a k-gate circuit, then the total approximation error is at most

$$\varepsilon$$

(Approximate universality)

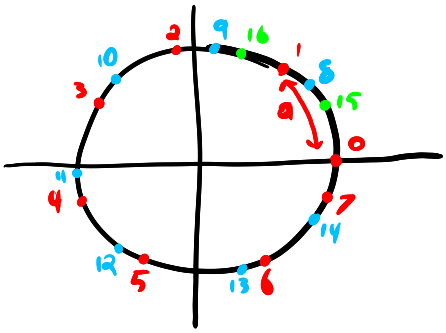
A set of gates Γ is approximately universal for quantum computing if any n-qubit unitary U can be approximated to arbitrary precision $\epsilon > 0$ by a circuit V over Γ. That is,

$$E(V, U) \leq \epsilon$$

Since the $\{CNOT\}$ U single-qubit gates set is exactly universal, it suffices to find a gate set which can approximate any single-qubit gate. The simplest way to do this is to write a single-qubit gate as

$$R_\ell(\alpha) R_k(\beta) R_\ell(\delta)$$

for some non-parallel axes ℓ & k, then use $R_\ell(a), R_k(b)$ where a & b are irrational multiples of π. This means if we keep rotating around axis ℓ at an angle of a, we never get back to an earlier spot



(Irrational rotations)

Recall that $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$. The gates

$$HTHT, \quad THTH$$

are irrational rotations around non-parallel axes

(Corollary)

The set $\underbrace{\{CNOT, H, T\}}$ is approximately universal.

called Clifford + T

(Efficiency of approximation)

Suppose we have an algorithm that runs on a quantum computer over the gate set CNOT + single-qubit unitaries in time $poly(n)$, but to approximate the circuit over $\{CNOT, H, T\}$ we might use $exp(n)$ gates! How can we be sure approximation will not incur exponential overhead? The seminal result of Solovay & Kitaev states that given certain assumptions (which have recently been proven unnecessary), approximations of single-qubit unitaries are efficient (polynomial).

(Solovay-Kitaev theorem)

Let Γ be a set of single qubit gates such that

1. Γ contains two rotations of angles which are irrational multiples of π around non-parallel axes, and
2. Γ is inverse-closed. That is, for every $U \in \Gamma$,

$$U^+ = U^{-1} \in \Gamma$$

Then any 1-qubit unitary may be approximated to error $\varepsilon > 0$ using $O(\log^c(1/\varepsilon))$ gates from Γ, where $c > 3$ for $\{H, T\}$.

While the approximation factor for $\{H, T\}$ has been improved to $O(\log(1/\varepsilon))$ by other methods in recent years, the Solovay-Kitaev theorem was instrumental in showing that quantum computation was practically possible. In particular, without the Solovay-Kitaev theorem, error correction and hence general purpose quantum computers was a pipe-dream...