CMPT 476 Lecture 17

More oracles!

Last class we discussed the black-box model and quantum query complexity with our first example of a truly quantum algorithm— Deutsch's algorithm. Today we continue with query algorithms, adding more complexity to our functions and the interference patterns leading to the desired answer. Just remember:

Quantum algorithms =

Superposition, interference, & entanglement

(Deutsch-Jozsa algorithm)

The next quantum algorithm we're going to see is a straightforward generalization of Deutsch's algorithm to the case when f takes n (rather than 1) inputs.

Let $f: \{0,1\}^n \longrightarrow \{0,1\}$. We say:

1. f is constant if $f(x) = f(y)$ $\forall x, y \in \{0,1\}^n$
2. f is balanced if $f(x) = 1$ for exactly half of the strings $x \in \{0,1\}^n$, and $f(x) = 0$ for the other half.

Deutsch-Josza's problem (DJ)

Input: a function $f: \{0,1\}^n \longrightarrow \{0,1\}$

Promise: f is either constant or balanced

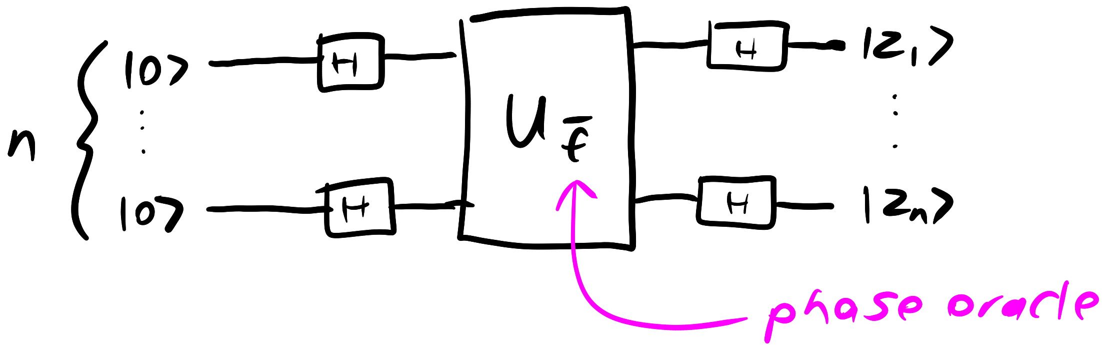
Goal: Determine whether f is constant or balanced

Fact: The classical query complexity is $2^{n-1}+1$

↳ Why? suppose the first 2^{n-1} queries (i.e. half the strings $x \in \{0,1\}^n$) give $f(x) = 0$. Then the other half of the strings could either all give 0 — hence f is constant — or could all give 1 — hence f is balanced.

Deutsch & Jozsa showed that the quantum query complexity of their problem is one!

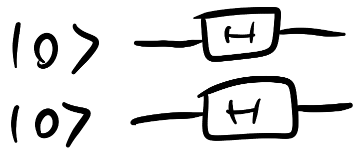
The Deutsch-Jozsa algorithm works analogously to Deutsch's algorithm, but with n qubits.



(Uniform superposition)

The first stage of the DJ algorithm is so common and important it deserves a separate analysis.

Consider the circuit



The state this circuit prepares is

$$(H|0\rangle)\otimes(H|0\rangle) = \tfrac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes\tfrac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$$
$$= \tfrac{1}{2}(|00\rangle+|01\rangle+|10\rangle+|11\rangle)$$
$$= \tfrac{1}{2}\sum_{x\in\{0,1\}^2}|x\rangle$$

This is a uniform superposition of $x\in\{0,1\}^2$.

In general,

$$H^{\otimes n}|0\rangle^{\otimes n} = \overbrace{(H\otimes H\otimes\cdots\otimes H)}^{n\text{ times}}\overbrace{|0\rangle|0\rangle\cdots|0\rangle}^{n\text{ times}}$$
$$=(H|0\rangle)^{\otimes n}$$
$$=\left(\tfrac{1}{\sqrt{2}}|0\rangle+\tfrac{1}{\sqrt{2}}|1\rangle\right)^{\otimes n}$$
$$= \frac{1}{\sqrt{2}^{\,n}}\sum_{x\in\{0,1\}^n}|x\rangle$$

So, Deutsch-Jozsa first prepares the uniform superposition then uses $U_f|x\rangle = (-1)^{f(x)}|x\rangle$ to phase each string:

$$U_f H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2}^n}\sum_{x\in\{0,1\}}(-1)^{f(x)}|x\rangle$$

As in the Deutsch algorithm, the final $H^{\otimes n}$ is going to generate interference.

But how?

(Hadamard gate, abstractly)

Note that $H|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle)$, $x\in\{0,1\}$

We can write this more compactly as

$$\begin{aligned} H|x\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle) \\ &= \frac{1}{\sqrt{2}}\sum_{z\in\{0,1\}}(-1)^{x\cdot z}|z\rangle \end{aligned}$$

Now what happens if we do this to an n-bit string?

$$\begin{aligned} H^{\otimes n}|x_1x_2\cdots x_n\rangle &= \left(\frac{1}{\sqrt{2}}\sum_{z_1}(-1)^{x_1\cdot z_1}|z_1\rangle\right)\otimes\cdots\otimes\left(\frac{1}{\sqrt{2}}\sum_{z_n}(-1)^{x_n\cdot z_n}|z_n\rangle\right) \\ &= \frac{1}{\sqrt{2}^n}\sum_{z_1\cdots z_n}(-1)^{x_1\cdot z_1+\cdots+x_n z_n}|z_1\cdots z_n\rangle \\ &= \frac{1}{\sqrt{2}^n}\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle \end{aligned}$$

$\leftarrow$ dot product over $\{0,1\}^n = \mathbb{Z}_2^n$

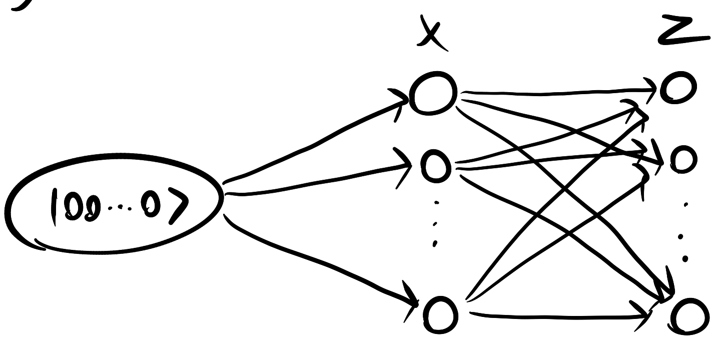
Note: we'll largely start using $\mathbb{Z}_2$ (integers mod 2) to refer to $\{0,1\}$ now. If $x,y\in\mathbb{Z}_2^n$,
$x\cdot y = x_1y_1\oplus\cdots\oplus x_ny_n$
$= x_1y_1+\cdots+x_ny_n \bmod 2$

So, the final state in the DJ algorithm is

$$\begin{aligned} &H^{\otimes n}\left(\frac{1}{\sqrt{2}^n}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}|x\rangle\right) \\ &= \frac{1}{\sqrt{2}^n}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}\left(\frac{1}{\sqrt{2}^n}\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle\right) \\ &= \frac{1}{2^n}\sum_{x,z\in\{0,1\}^n}(-1)^{f(x)+x\cdot z}|z\rangle \end{aligned}$$

(Interference analysis)

The algorithm looks like this:



We need to figure out which paths interefere.

Consider a single z. The amplitude of this z is the sum over all paths leading to it:

$$\frac{1}{2^n} \sum_{x \in \{0,1\}} (-1)^{f(x)+x\cdot z} |z\rangle$$

What is the amplitude of $z = 00\cdots0$?

Case 1: f is constant

Then $\frac{1}{2^n} \sum_{x\in\{0,1\}} (-1)^{f(x)} |00\cdots0\rangle = \frac{1}{2^n} \sum_x (-1)^c |00\cdots0\rangle$

$= \pm |00\cdots0\rangle$

Case 2: f is balanced

The $\frac{1}{2^n} \sum_x (-1)^{f(x)} |00\cdots0\rangle = \frac{1}{2^n}\left(\sum_{x|f(x)=0} |00\cdots0\rangle + \sum_{x|f(x)=1} -|00\cdots0\rangle\right)$

$= \frac{2^{n-1}}{2^n} |00\cdots0\rangle - \frac{2^{n-1}}{2^n} |00\cdots0\rangle$

$= 0$

So, if we measure at the end, if f is constant we get $|00\cdots0\rangle$ with 100% probability, and if f is balanced we got $|00\cdots0\rangle$ with 0% probability!

(Bernstein-Vazirani algorithm)

The Deutsch-Jozsa algorithm is not that impressive in reality, because we can solve the problem with 2/3 probability with 2 queries classically using a randomized algorithm. Bernstein & Vazirani came up with the next algorithm that gives a non-trivial speed-up over randomized algorithms too!
Their algorithm is identical to Deutsch-Jozsa, but involves a specially-chosen promise on f.

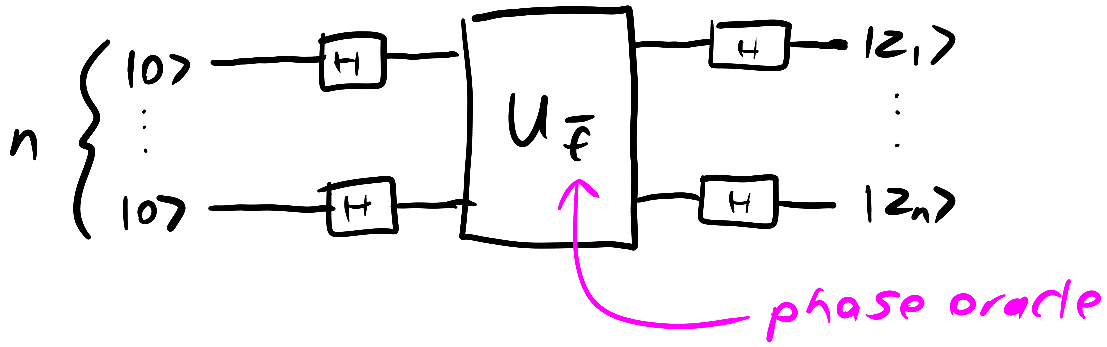
Bernstein-Vazirani problem (BV)

Input: a function $f: \{0,1\}^n \to \{0,1\}$
Promise: $f(x) = s \cdot x \mod 2 \quad \forall x \in \{0,1\}^n$ for some $s \in \{0,1\}^n$
Goal: find the hidden string s

Fact

The probabilistic query complexity of BV is at least n. Why? Because we need n bits of information and f only gives us 1 bit.

Bernstein & Vazirani's algorithm uses the exact same circuit as Deutsch & Jozsa's, but a different interference analysis



Final state: $\frac{1}{2^n} \sum_{x,z \in \{0,1\}^n} (-1)^{f(x) + x \cdot z} |z\rangle$

(Interference analysis)

The simple analysis is, just like Deutsch-Jozsa, to look at the amplitude of a well-chosen string. This time, we'll analyze interference when $z = s$.

$$\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x) + x \cdot s} |s\rangle = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{s \cdot x + x \cdot s} |s\rangle$$

$$= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} |s\rangle$$

$$= |s\rangle$$

So measuring in the computational basis results in s with 100% probability!

Simple, right?