SIMON FRASER UNIVERSITY
School of Computing Science

CMPT 476/981– FINAL EXAM
Introduction to Quantum Algorithms

Instructor: Matt Amy

2024/05/01

Name: ______________________________
Student Number: ______________________________

Instructions:

- **1 double-sided sheet of 8.5x11" paper is permitted as a cheat-sheet**
- A **non-programmable** calculator is permitted
- **No other aids are permitted**
- Print your **full name** and **student ID number** in the space above
- There are 14 pages including this cover page and 10 questions
- The total number of points is 58.
- You will have **180** minutes
- **Good luck!**

Distribution of Marks

Question	Points	Score
1	10	
2	4	
3	6	
4	4	
5	5	
6	4	
7	5	
8	5	
9	10	
10	5	
Total:	58	

1. (10 points) Quantum trivia, 1 point each

(a) Give one way we can perform computation on quantum states other than by using unitary transformations.

(b) Circle the ground state energy of the following Hamiltonian:

$$\begin{bmatrix} 0.707 & 0 & 0 & 0 \\ 0 & 0.235 & 0 & 0 \\ 0 & 0 & 0.674 & 0 \\ 0 & 0 & 0 & 0.125 \end{bmatrix}$$

(c) Which is computationally harder for a quantum computer: computing a random eigenvalue, or the ground state energy of a Hamiltonian?

(d) List **3** ingredients necessary for a quantum speed-up over classical computation.

(e) How many **physical** and **logical** qubits are in the code below?

$$|00\rangle_L = |11111111\rangle \qquad |01\rangle_L = |01010101\rangle$$
$$|10\rangle_L = |00110011\rangle \qquad |11\rangle_L = |00001111\rangle$$

(f) In the code above, how many **bit** flips can be successfully corrected?

(g) Given a code encoding 1 logical qubit into 5 physical qubits, how many physical physics are used to encode 1 logical qubit if we concatenate this code 3 times?

(h) What do we call a process that maps **density matrices** to **density matrices**?

(i) In the query complexity model, which algorithm first gave an **constant-factor** speed-up over **non-probabilitistic** classical algorithms? (circle one)

Deutsch **Deutsch-Josza** **Bernstein-Vazirani** **Simon**

(j) In the query complexity model, which algorithm first gave an **linear** speed-up over **probabilitistic** classical algorithms?

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2. (4 points) Show that the 3-qubit state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

cannot be written as a tensor product of 1-qubit states, i.e.

$$|GHZ\rangle \neq |\psi\rangle \otimes |\varphi\rangle \otimes |\theta\rangle.$$

3. Let

$$|\psi\rangle = a|000\rangle + b|001\rangle + c|010\rangle + d|011\rangle + e|100\rangle + f|101\rangle + g|110\rangle + h|111\rangle$$

be the general state of a three-qubit system.

(a) (4 points) Give the probabilities and the resulting state if the **second** qubit of $|\psi\rangle$ is measured in the computational basis. Explicitly, give

1. the probabilities of both outcomes (0 and 1), **and**
2. the resulting state associated with each outcome.

(b) (2 points) Calculate the probabilities of measuring 0 and 1 in the **second** qubit

$$|\psi\rangle = \frac{1}{\sqrt{8}}|000\rangle + \frac{\sqrt{3}}{\sqrt{8}}|010\rangle + \frac{i\sqrt{2}}{\sqrt{8}}|011\rangle + \frac{1}{\sqrt{8}}|100\rangle + \frac{-i}{\sqrt{8}}|111\rangle.$$

4. (4 points) Prove that for any complex-valued matrices A, B, $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

5. (5 points) Devise a reversible circuit composed of X, $CNOT$, and Toffoli gates computing the following function:

$$f(x_1, x_2, x_3, x_4) = (x_1 \wedge (x_2 \wedge (x_3 \wedge x_4))) \oplus (\neg x_1 \wedge (x_2 \wedge x_3)) \oplus (x_1 \wedge (x_2 \oplus (x_3 \oplus x_4)))$$

Your circuit should uncompute any temporary/intermediate values it uses.

6. (4 points) Let $f : \{0,1\} \to \{0,1\}$ be some Boolean function. Show that

$$\frac{1}{2\sqrt{2}} \sum_{x,y,z \in \{0,1\}} (-1)^{f(x)y + yz} |z\rangle = \frac{1}{\sqrt{2}} \sum_{x \in \{0,1\}} |f(x)\rangle$$

7. (5 points) Let $|s\rangle = \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}|x\rangle$ and suppose $|s\rangle$ can be written as

$$|s\rangle = \cos(\theta)|\psi_{bad}\rangle + \sin(\theta)|\psi_{good}\rangle$$

where $|\psi_{bad}\rangle$ and $|\psi_{good}\rangle$ are **orthogonal** unit vectors. Show that the Grover iterate $Q = U_{diff}U_f$ where $U_f = I - 2|\psi_{good}\rangle\langle\psi_{good}|$ and $U_{diff} = 2|s\rangle\langle s| - I$ implements the rotation matrix

$$Q = \begin{bmatrix}\cos(2\theta) & -\sin(2\theta)\\ \sin(2\theta) & \cos(2\theta)\end{bmatrix}$$

on the two-dimensional subspace $\text{span}(\{|\psi_{bad}\rangle, |\psi_{good}\rangle\})$, where a superposition $a|\psi_{bad}\rangle + b|\psi_{good}\rangle$ corresponds to the vector $\begin{bmatrix}a\\ b\end{bmatrix}$. You may find the following trig identities helpful:

- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $1 = \cos^2(\theta) + \sin^2(\theta)$

8. (a) (3 points) Write out the 5×5 matrix for a Quantum Fourier Transform on a 5-dimensional state — that is,

$$QFT_5|x\rangle = \frac{1}{\sqrt{5}} \sum_{y \in \mathbb{Z}_5} \omega^{xy} |y\rangle$$

where xy is interpreted as integer multiplication mod 5 and $\omega = e^{\frac{2\pi i}{5}}$ is an 5th root of unity. **Note that the QFT is equal to its transpose, so you only need to calculate the entries above the diagonal**

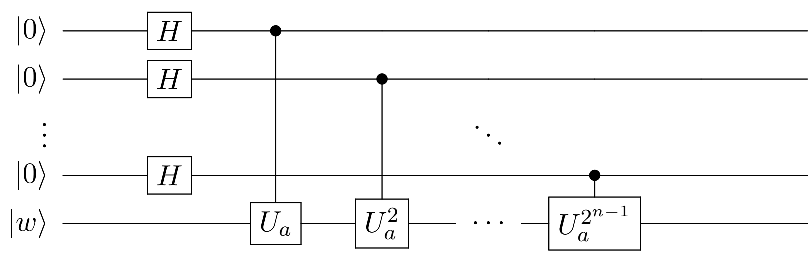
(b) (2 points) Give an explicit circuit for the **3-qubit** ($2^3 = 8$-dimensional) Quantum Fourier Transform

9. Let $U_a|x\rangle = |ax \mod M\rangle$ where $a, x \in \mathbb{Z}_{2^n} = \{0, 1, \ldots, 2^n - 1\}$ be an oracle for multiplication mod M of n-bit integers $x = x_0 + 2x_1 + \cdots + 2^{n-1}x_{n-1}$, and suppose $U_a^r = I$.

   (a) (2 points) Show that for any eigenvalue λ of U_a, $\lambda^r = 1$ and hence λ is an r-th root of unity.

   (b) (5 points) Let $U_a|w\rangle = e^{2\pi i \frac{1}{r}}|w\rangle$. Show that the circuit below produces the state

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \mathbb{Z}_{2^n}} e^{2\pi i \frac{x}{r}} |x\rangle|w\rangle$$



(c) (3 points) Suppose $r = 2^m$ for some $m < n$. Show that

$$QFT_{2^n}^{-1}\left(\frac{1}{\sqrt{2^n}}\sum_{x\in\mathbb{Z}_{2^n}} e^{2\pi i\frac{x}{r}}|x\rangle\right) = |2^{n-m}\rangle$$

10. (5 points) Suppose you're given a polynomial-time quantum algorithm which is exponentially faster than the best-known classical algorithm for the same problem — however, the algorithm is guaranteed to **never produce destructive interference** (i.e. paths leading to the same state always have the same phase). Assuming it's implemented over the universal gate set $\{H, \text{Toffoli}\}$, describe how you can use the quantum algorithm to give a **polynomial-time probabilistic algorithm** with the same success probability as the quantum version for the problem.

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