CMPT 476/776: Introduction to Quantum Algorithms
Assignment 0 — Mathematical preliminaries

Not graded.

Question 1: Complex numbers

1. Compute the following in standard form :
   - $(2+3i)+(1-4i)$
   - $(1+2i)(3-i)$
   - $|4-3i|$
   - $\frac{2+i}{1-3i}$
2. Compute the following in polar form :
   - $4-3i$
   - $(2e^{i\pi/4})(5e^{i\pi/8})$
   - $(3e^{i\pi/4})^2$
   - $\frac{1}{7e^{i3\pi/5}}$
3. Verify that $z+z^* = 2\mathrm{Re}(z)$
4. Verify that $z-z^* = 2i\mathrm{Im}(z)$
5. Verify that $zz^* = |z|^2$

Solution.

1.
   - $(2+3i)+(1-4i) = (2+1)+(3-4)i = 3-i.$
   - $(1+2i)(3-i) = 3-i+6i-2i^2 = 3+5i+2 = 5+5i.$
   - $|4-3i| = \sqrt{4^2+(-3)^2} = \sqrt{16+9} = \sqrt{25} = 5.$
   - $\frac{2+i}{1-3i} = \frac{2+i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{(2+i)(1+3i)}{(1-3i)(1+3i)} = \frac{2+6i+i+3i^2}{1-9i^2} = \frac{-1+7i}{10} = -0.1+0.7i.$
2.
   - $4-3i = re^{i\theta} = r(\cos(\theta)+i\sin(\theta)) = 5(\frac{4}{5}-\frac{3}{5}i) = 5e^{-i\cos^{-1}(0.8)}.$
   - $(2e^{i\pi/4})(5e^{i\pi/8}) = 10e^{i\pi/4+i\pi/8} = 10e^{i3\pi/8}.$
   - $(3e^{i\pi/4})^2 = 3^2(e^{i\pi/4})^2 = 9e^{i\pi/2}.$
   - $\frac{1}{7e^{i3\pi/5}} = \frac{1}{7}(e^{i3\pi/5})^{-1} = \frac{1}{7}e^{-i3\pi/5} = \frac{1}{7}e^{i7\pi/5}.$

3. We know that $z = a + bi$, where $a, b \in \mathbb{R}$ and $\mathrm{Re}(z) = a$.
   Hence, $z + z^* = (a + bi) + (a - bi) = (a + a) + (b - b)i = 2a + 0 = 2a = 2\mathrm{Re}(z)$.

4. We know that $z = a + bi$, where $a, b \in \mathbb{R}$ and $\mathrm{Im}(z) = b$.
   Hence, $z - z^* = (a + bi) - (a - bi) = (a - a) + (b - (-b))i = 0 + 2b = 2b = 2\mathrm{Im}(z)$.

5. We know that $z = a + bi$, where $a, b \in \mathbb{R}$ and $|z| = \sqrt{a^2 + b^2}$.
   Hence, $zz^* = (a + bi)(a - bi) = a^2 - abi + bai - b^2i^2 = a^2 + (-ab + ab)i + b^2 = a^2 + 0 + b^2 = (\sqrt{a^2 + b^2})^2 = |z|^2$.

Question 2: Linear algebra

1. Let $\mathbf{v} = \begin{bmatrix} 1+i \\ 2-i \\ -i \end{bmatrix}$.

   (a) Compute the norm of $\mathbf{v}$.

   (b) Normalize $\mathbf{v}$ to obtain a unit vector.

2. Compute $A\mathbf{v}$ where $A = \begin{bmatrix} 2 & i & 1 \\ -i & 3 & -i \\ 1 & i & 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -i \\ 2 \end{bmatrix}$.

3. Write the matrix $A = \begin{bmatrix} 2 & i & 1 \\ -i & 3 & -i \\ 1 & i & 4 \end{bmatrix}$ in reduced Echelon form via Gaussian elimination.

4. Consider the matrix $B = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

   (a) Find the eigenvalues of B.

   (b) Find the eigenvectors corresponding to each eigenvalue and normalize them.

5. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1+i \\ 1+i \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$.

   (a) Write each vector as a linear combination over the standard basis of $\mathbb{C}^3$:

   $$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

   (b) Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form a basis for $\mathbb{C}^3$?

   (c) What is the dimension of the subspace spanned by $\{\mathbf{v}_2, \mathbf{v}_3\}$? What about just $\{\mathbf{v}_3\}$?

Solution.

1. (a) $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{\sum_i |v_i|^2} = \sqrt{|1+i|^2 + |2-i|^2 + |-i|^2} = \sqrt{2+5+1} = \sqrt{8}.$

(b) $\mathbf{N_v} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{1}{\sqrt{8}} \begin{bmatrix} 1+i \\ 2-i \\ -i \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{8}} + \frac{i}{\sqrt{8}} \\ \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{8}} \\ \frac{-i}{\sqrt{8}} \end{bmatrix}.$

2. $A\mathbf{v} = \begin{bmatrix} 2-i^2+2 \\ -i-3i-2i \\ 1-i^2+8 \end{bmatrix} = \begin{bmatrix} 5 \\ -6i \\ 10 \end{bmatrix}.$

3. Starting with $A = \begin{bmatrix} 2 & i & 1 \\ -i & 3 & -i \\ 1 & i & 4 \end{bmatrix}$, we first do forward elimination:

$R_1 \leftrightarrow R_3 : \begin{bmatrix} 1 & i & 4 \\ -i & 3 & -i \\ 2 & i & 1 \end{bmatrix}$, $iR_1 + R_2 \to R_2 : \begin{bmatrix} 1 & i & 4 \\ 0 & 2 & 3i \\ 2 & i & 1 \end{bmatrix}$, $-2R_1 + R_3 \to R_3 : \begin{bmatrix} 1 & i & 4 \\ 0 & 2 & 3i \\ 0 & -i & -7 \end{bmatrix}$,

$R_2 \leftrightarrow R_3 : \begin{bmatrix} 1 & i & 4 \\ 0 & -i & -7 \\ 0 & 2 & 3i \end{bmatrix}$, $iR_2 \to R_2 : \begin{bmatrix} 1 & i & 4 \\ 0 & 1 & -7i \\ 0 & 2 & 3i \end{bmatrix}$, $-2R_2 + R_3 \to R_3 : \begin{bmatrix} 1 & i & 4 \\ 0 & 1 & -7i \\ 0 & 0 & 17i \end{bmatrix}$,

$\frac{-i}{17} R_3 \to R_3 : \begin{bmatrix} 1 & i & 4 \\ 0 & 1 & -7i \\ 0 & 0 & 1 \end{bmatrix}$;

Now that we obtained the Echelon form of the matrix, we do backward substitution:

$7iR_3 + R_2 \to R_2 : \begin{bmatrix} 1 & i & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $-4R_3 + R_1 \to R_1 : \begin{bmatrix} 1 & i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$-iR_2 + R_1 \to R_1 : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which is in the reduced Echelon form.

4. (a) A classic way of finding the eigenvalues of a matrix is to find the roots of its characteristic polynomial, $\det(B - \lambda I) = \begin{vmatrix} -\lambda & -i & 0 \\ i & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 1) + i(-i\lambda) = -\lambda(\lambda^2 - 2).$

So the eigenvalues are $\lambda \in \{0, \pm\sqrt{2}\}$.

(b) $\lambda_1 = 0 : B\mathbf{v}_1 = 0 \Rightarrow \begin{bmatrix} -iv_{12} \\ iv_{11} + v_{13} \\ v_{12} \end{bmatrix} = 0 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} \Rightarrow ||\mathbf{v}_1|| = \sqrt{2} \Rightarrow \mathbf{N_{v_1}} = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}$,

$\lambda_2 = \sqrt{2} : (B - \sqrt{2}I)\mathbf{v}_2 = 0 \Rightarrow \begin{bmatrix} -\sqrt{2}v_{21} - iv_{22} \\ iv_{21} - \sqrt{2}v_{22} + v_{23} \\ v_{22} - \sqrt{2}v_{23} \end{bmatrix} = 0 \Rightarrow \mathbf{v}_2 = \begin{bmatrix} -i \\ \sqrt{2} \\ 1 \end{bmatrix} \Rightarrow ||\mathbf{v}_2|| = 2 \Rightarrow$

$\mathbf{N_{v_2}} = \frac{1}{2} \begin{bmatrix} -i \\ \sqrt{2} \\ 1 \end{bmatrix}$,

$\lambda_3 = -\sqrt{2} : (B + \sqrt{2}I)\mathbf{v}_3 = 0 \Rightarrow \begin{bmatrix} \sqrt{2}v_{31} - iv_{32} \\ iv_{31} + \sqrt{2}v_{32} + v_{33} \\ v_{32} + \sqrt{2}v_{33} \end{bmatrix} = 0 \Rightarrow \mathbf{v}_3 = \begin{bmatrix} -i \\ -\sqrt{2} \\ 1 \end{bmatrix} \Rightarrow ||\mathbf{v}_3|| = 2 \Rightarrow \mathbf{N}_{\mathbf{v_3}} = \frac{1}{2}\begin{bmatrix} -i \\ -\sqrt{2} \\ 1 \end{bmatrix}$.

5. (a) $\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_1 + (1+i)\mathbf{e}_2 + (1+i)\mathbf{e}_3$, and $\mathbf{v}_3 = \mathbf{e}_1 + 5\mathbf{e}_2 + 5\mathbf{e}_3$.
   (b) No, since $(-4+i)\mathbf{v}_1 + 5\mathbf{v}_2 - (1+i)\mathbf{v}_3 = \mathbf{0}$, i.e. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not linearly independent.
   (c) Since for all $a_2, a_3 \in \mathbb{C}$, $a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = 0 \Rightarrow a_2 = a_3 = 0$, the vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent. Therefore, they form a basis for the subspace they span, and $\dim(\mathrm{span}(\{\mathbf{v}_2, \mathbf{v}_3\})) = 2$.
   The subspace spanned by $\{\mathbf{v}_3\}$ has dimension 1, with $\mathbf{v}_3$ as a basis.

Question 3: Probability theory

1. A box contains 3 red balls and 2 blue balls. Two balls are drawn at random without replacement.
   (a) What is the probability that the first ball drawn is red?
   (b) Given that the first ball drawn is red, what is the probability that the second ball drawn is also red?
   (c) What is the probability that both balls drawn are red?
2. A discrete random variable X represents the number rolled on a fair six-sided die. Compute the expected value of X.
3. A factory produces items from three machines: Machine A, Machine B, and Machine C.
   - 50% of the items are from Machine A, 30% from Machine B, and 20% from Machine C.
   - The probabilities of a defective item are 2%, 5%, and 10% for Machines A, B, and C, respectively.

   (a) What is the probability that a randomly selected item is defective?
   (b) If an item is found to be defective, what is the probability it was produced by Machine B?
4. A test for a disease has the following characteristics:
   - 99% of people with the disease test positive.
   - 95% of people without the disease test negative.
   - 1% of the population has the disease.

   If a person tests positive, what is the probability they actually have the disease? Use Baye's theorem,
   $$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Solution.

1. (a) $\frac{3}{3+2} = \frac{3}{5}$.
   (b) $\frac{2}{4} = \frac{1}{2}$.
   (c) $\frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$.

2. $\mathbb{E}[X] = \frac{1+2+3+4+5+6}{6} = 3.5$.

3. (a) $0.5 \times 0.02 + 0.3 \times 0.05 + 0.2 \times 0.1 = 0.045 = 4.5\%$.
   (b) $P(B|\text{defective}) = \frac{P(B \text{ and defective})}{P(\text{defective})} = \frac{0.3 \times 0.05}{0.045} = \frac{1}{3}$.

4. Let T be the event that someone tests positive for the disease. Let D be the event that someone has the disease. We want to calculate

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)}$$

We are given $P(T|D) = 0.99$, $P(\overline{T}|\overline{D}) = 0.95$, $P(D) = 0.01$. Thus

$$\begin{aligned} P(T) &= P(T|D) \cdot P(D) + P(T|\overline{D}) \cdot P(\overline{D}) \\ &= P(T|D) \cdot P(D) + (1 - P(\overline{T}|\overline{D})) \cdot P(\overline{D}) \\ &= 0.99 \times 0.01 + (1 - 0.95)(1 - 0.01) \\ &= 0.0594 \end{aligned}$$

Hence

$$P(D|T) = \frac{0.99 \times 0.01}{0.0594} = \frac{1}{6} = 16.67\%.$$

Question 4: Discrete math

1. Compute the following:
   - $27 \mod 5$,
   - $123 \mod 17$,
   - $-11 \mod 7$.

2. Solve $3x \equiv 1 \pmod 5$ for x.

3. Factorize 84 into its prime components.

4. Compute the greatest common divisor (GCD) of 24 and 36

5. Compute the least common multiple (LCM) of 24 and 36

Solution.

1. - $27 \mod 5 \equiv 2 \mod 5$
   - $123 \mod 17 \equiv 4 \mod 17$

- $-11 \mod 7 \equiv (-11+14) \mod 7 \equiv 3 \mod 7$

2. $3 \cdot 2 = 6 \pmod 5 \equiv 1 \pmod 5 \implies x = 2 \pmod 5$.
3. $84 = 2^2 \cdot 3 \cdot 7$.
4. $24 = 2^3 \cdot 3$, $36 = 2^2 \cdot 3^2$. Hence $\gcd(24, 36) = 2^2 \cdot 3 = 12$.
5. $\text{lcm}(24, 36) = 2^3 \cdot 3^2 = 72$.