CMPT 476/776: Introduction to Quantum Algorithms
Assignment 1

Due **January 22, 2026 at 11:59pm on Crowdmark**
Complete individually and submit in PDF format.

Question 1 [6 points]: Classical circuits

The $NAND$ gate is a classical gate with the following truth table:

x	y	$NAND(x,y)$
0	0	1
0	1	1
1	0	1
1	1	0

1. Show by induction on n that the gate set $\{AND, OR, NOT, FANOUT\}$ can implement any function $f : \{0,1\}^n \to \{0,1\}$. **You may find the following identity helpful:**

$$f(x_1, \ldots, x_{n-1}, x_n) = (x_n \wedge f(x_1, \ldots, x_{n-1}, 1)) \vee (\neg x_n \wedge f(x_1, \ldots, x_{n-1}, 0)).$$

2. Show that the gate set $\{NAND, FANOUT\}$ is universal for classical computation by giving implementations of each gate of the universal set $\{AND, OR, NOT, FANOUT\}$.

3. Suppose we encode the state of two classical bits $x, y \in \{0,1\}$ as 4-dimensional vectors labelled $|x, y\rangle$ with the following encoding:

$$|0,0\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad |0,1\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad |1,0\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad |1,1\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

   Give a 4x4 matrix which maps the vector $|x, y\rangle$ to $|x, NAND(x,y)\rangle$ for any $x, y \in \{0,1\}$.

4. Is the matrix you gave in part 2 invertible?

Question 2 [4 points]: Dirac notation

Let $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{-i}{\sqrt{2}}|2\rangle$, $|\phi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{i}{\sqrt{3}}|1\rangle + \frac{-1}{\sqrt{3}}|2\rangle$ be two states of a **qutrit** (i.e. a three-level or three-dimensional system).

1. Write $|\psi\rangle$ and $|\phi\rangle$ explicitly as column vectors
2. Calculate the following:
   - $\langle\psi|\psi\rangle$
   - $\langle\psi|\phi\rangle$
   - $|\psi\rangle\langle\phi|$
   - $|\psi\rangle \otimes |\phi\rangle$
3. Is the vector $|\psi\rangle + |\phi\rangle$ a unit vector? If not, normalize it to get a unit vector.

Question 3 [6 points]: Qubits, gates, and measurement

Suppose we have a qubit initially in the state $\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle$ for some $\theta \in \mathbb{R}$.

1. Calculate the probabilities of receiving result "0" or "1" if the qubit is measured.
2. Recall the definition of the Hadamard gate, which is the change of basis matrix for the $\{|+\rangle, |-\rangle\}$ basis:
$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
   If we first apply the Hadamard gate to the initial state $\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle$ and then measure in the computational basis, what are the probabilities of receiving the "0" and "1" results as **(simplified)** functions of θ?
3. Using computational basis measurement, H gates, and *phase gates*
$$P(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$
   for any $\theta \in \mathbb{R}$, give a protocol to distinguish with 100% accuracy between the states
$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{-i\pi/4}|1\rangle), \qquad |\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i3\pi/4}|1\rangle)$$
4. Suppose you are given k identical copies of the state $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle$ where $\theta \in [0, \pi]$ is unknown. Using only H gates and computational basis measurements, give a procedure to estimate the value of θ. Your method should converge to the correct value of θ as $k \to \infty$.

Question 4 [4 points]: Eigenvectors

1. Let $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. Find two **unit** vectors $|+_Y\rangle$, $|-_Y\rangle$ such that

$$Y|+_Y\rangle = |+_Y\rangle$$
$$Y|-_Y\rangle = -|-_Y\rangle$$

2. Let U be the 2 by 2 matrix with columns $|+_Y\rangle$ and $|-_Y\rangle$. Is U unitary?

3. Calculate $U^\dagger Y U$. What do you notice?