SIMON FRASER UNIVERSITY
School of Computing Science

CMPT 476/981– MIDTERM EXAM
Introduction to Quantum Algorithms

Instructor: Matt Amy

2024/02/29

Name: ____
Student Number: ______________________________________

Instructions:

- **1 double-sided sheet of 8.5x11" paper is permitted as a cheat-sheet**
- A **non-programmable** calculator is permitted
- **No other aids are permitted**
- Print your **full name** and **student ID number** in the space above
- There are 10 pages including this cover page and 8 questions
- The total number of points is 46.
- You will have **110** minutes
- **Good luck!**

Distribution of Marks

Question	Points	Score
1	10	
2	7	
3	7	
4	6	
5	5	
6	2	
7	4	
8	5	
Total:	46	

1. (10 points) Short answers, 1 point each

   (a) What is the dimension of the state space of n qubits?

   (b) What is the definition of a unitary operator (you do not need to define the dagger $(\cdot)^\dagger$)

   (c) What is the maximum number of dimensions a single particle's quantum state can have?

   (d) What is the probability of measuring $|0\rangle$ in the state $a|0\rangle + b|1\rangle + c|2\rangle$?

   (e) Write the state $a|0\rangle + b|1\rangle + c|2\rangle$ as a vector.

   (f) Give one way in which quantum computation is different from probabilistic computation.

   (g) Normalize the vector $\sqrt{5}|0\rangle + \sqrt{-11}|1\rangle$

   (h) Complete the expression:

   $$e^{i\theta} = \cos\theta + \text{\_\_\_\_\_\_\_\_}$$

   (i) Complete the expression:

   $$\mathrm{Tr}\left(\begin{bmatrix} 3 & 5 & 1 \\ 0 & 4 & 9 \\ 5 & 5 & 5 \end{bmatrix}\right) = \text{\_\_\_\_\_\_\_\_}$$

   (j) Give one way a (quantum) controlled gate $c - U$ is different from a classically controlled gate U^x — i.e. applying a gate depending on the value of a classical bit $x \in \{0, 1\}$.

2. (a) (3 points) Calculate the probabilities of obtaining each result when measuring the state

$$|\psi\rangle = \frac{3}{5}|0\rangle + \frac{-4i}{5}|1\rangle$$

in the basis

$$\{|A\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |B\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\}$$

(b) (4 points) Calculate the probabilities and give the corresponding final state if the first qubit of the state

$$|\psi\rangle = \frac{1}{5}|00\rangle + \frac{\sqrt{3}}{5}|01\rangle + \frac{1}{\sqrt{5}}|10\rangle + \frac{4}{5}|11\rangle$$

is measured in the computational basis.

3. (a) (2 points) Write the following 3-qubit state as a linear combination over the **3-qubit computational (binary) basis**:

$$\frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ i \\ -1 \\ 0 \\ -i \\ 0 \end{bmatrix}$$

(b) (3 points) Write the following 2-qubit operator as a linear combination over the computational basis of 2-qubit operators $\{|ij\rangle\langle lk| \mid i, j, l, k \in \{0, 1\}\}$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & -i\sqrt{2} \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i\sqrt{2} & 0 & 0 & 0 \end{bmatrix}$$

(c) (2 points) To combat noise and decoherence, we often encode a **logical** qubit inside a **subspace** of a larger Hilbert space. Suppose we use the subspace span($\{|00\rangle, |11\rangle\}$) of $\mathbb{C}^2 \otimes \mathbb{C}^2$ to encode one logical qubit, where the "0" state is taken to be $|00\rangle$ and the "1" state is $|11\rangle$. What 1 qubit gate (i.e. a 2×2 unitary transformation) does the operator in the previous part perform **on this subspace?**

4. (a) (2 points) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} e & f \\ g & h \\ i & j \end{bmatrix}.$$

Write down the matrix $A \otimes B$.

(b) (4 points) Let $\mathbb{C}^{m\times n}$ denote the space of complex-valued matrices with m rows and n columns, and define the following constants:

$$\begin{array}{lll} |\psi\rangle \in \mathbb{C}^2 = \mathbb{C}^{2\times 1} & A \in \mathbb{C}^{2\times 2} & E \in \mathbb{C}^{4\times 8} \\ |\phi\rangle \in \mathbb{C}^8 = \mathbb{C}^{8\times 1} & B \in \mathbb{C}^{8\times 8} & I \in \mathbb{C}^{2\times 2} \\ |\Delta\rangle \in \mathbb{C}^4 = \mathbb{C}^{4\times 1} & C \in \mathbb{C}^{4\times 4} & \\ |\zeta\rangle \in \mathbb{C}^4 = \mathbb{C}^{4\times 1} & D \in \mathbb{C}^{4\times 4} & \end{array}$$

Correctly parenthesize the expression below to make it well-formed, keeping in mind that

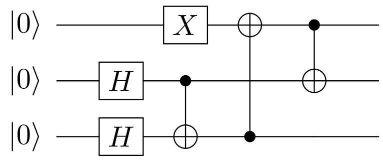
- $A + B$ is well formed if and only if the dimensions of A and B are equal, and
- AB is well-formed if and only if the columns of A equal the rows of B.

Hint: work from right to left

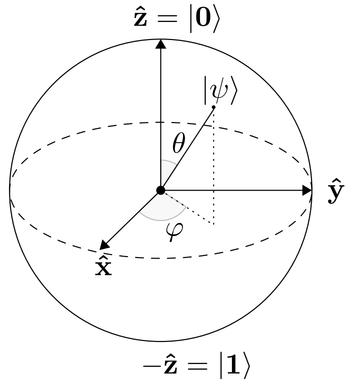
$$\langle\zeta| \otimes I \cdot E \otimes I \cdot C + |\Delta\rangle\langle\Delta| \otimes D \cdot B \otimes I \cdot |\phi\rangle \otimes A \cdot |\psi\rangle$$

(c) (1 point (bonus)) Draw the expression in part (b) as a circuit diagram.

5. (5 points) Calculate the final state of the circuit below in the computational basis.



6. (2 points) Recall that in the Bloch sphere, a qubit has state $|\psi\rangle = \cos(\frac{\theta}{2})|0\rangle + e^{i\varphi}\sin(\frac{\theta}{2})|1\rangle$ where θ is the angle the state makes with the positive z-axis and ϕ the angle it makes with the positive x-axis.



Implement a transformation that maps the $|0\rangle$ state to any point $\cos(\frac{\theta}{2})|0\rangle + e^{i\varphi}\sin(\frac{\theta}{2})|1\rangle$ on the Bloch sphere using rotations around the x-, y-, and/or z-axes. Recall that the corresponding rotation matrices are defined as

$$R_x(\theta) = \begin{bmatrix} \cos(\frac{\theta}{2}) & i\sin(\frac{\theta}{2}) \\ -i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix}, \qquad R_y(\theta) = \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix}, \qquad R_z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

7. (4 points) Using $CNOT$ and H gates and computational-basis measurement, give a procedure to distinguish with **100% accuracy** between the following states

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1100\rangle + |0011\rangle), \qquad |\phi\rangle = \frac{1}{\sqrt{2}}(|1100\rangle - |0011\rangle)$$

8. Suppose Alice and Bob share an EPR pair $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Suppose Bob applies a unitary U to his qubit and then keeps it in storage.

   (a) (4 points) Suppose a year has passed and Alice creates some qubit in the state $|\psi\rangle$. She then measures this qubit with her half of the EPR pair in the Bell basis to teleport it to Bob, and obtains measurement result β_{00}. What is the resulting state of Bob's qubit? If it helps, recall the definition of the Bell basis:

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \qquad |\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

   (b) (1 point) Argue whether or not the above could be considered to be a violation of **causality** — the notion that causes and effect occur in the order in which they happen.

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