

Exercises on Functions. Complete by: Thursday, July 9th at 11:59pm

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1. Is the function $f: \mathbb{Z} \rightarrow \mathbb{Z}$ defined as $f(n) = n^2 + 2$ one-to-one?
2. Determine whether or not the function $f: \mathbb{Z} \times (\mathbb{Z} - \{0\}) \rightarrow \mathbb{Z}$ is onto, if $f((m, n)) = \lfloor \frac{m}{n} \rfloor$.
3. Let f be a function from X to Y . For $A \subseteq X$, let also $f(A)$ denote the set $f(A) = \{f(a) \mid a \in A\}$. Prove that f is one-to-one if and only if

$$f(A \cap B) = f(A) \cap f(B)$$

for any subsets $A, B \subseteq X$.

4. Let f be a function from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z} \times \mathbb{Z}$ (that is, f maps pairs of integers to pairs of integers) given by $f(x, y) = (x + y, x - y)$. Is f one-to-one? onto? Prove or give a counterexample.
5. Given $g = \{(1, b), (2, c), (3, a)\}$, a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$, and $f = \{(a, \gamma), (b, \alpha), (c, \delta), (d, \gamma)\}$, a function from Y to $Z = \{\alpha, \beta, \gamma, \delta\}$, write $f \circ g$ as a set of ordered pairs.
6. Let $f(x) = ax + b$ and $g(x) = cx^2 + dx$, where $a, b, c,$ and d are constants. Compute $f \circ g$ and $g \circ f$. Determine for which constants $a, b, c,$ and d it is true that $f \circ g = g \circ f$.
(Hint: Note that polynomials $d_n x^n + d_{n-1} x^{n-1} + \dots + d_1 x + d_0$ and $e_n x^n + e_{n-1} x^{n-1} + \dots + e_1 x + e_0$ are equal as functions if and only if $d_n = e_n, d_{n-1} = e_{n-1}, \dots, d_1 = e_1, d_0 = e_0$.)
7. Let f, g be functions from positive integers to positive integers given by formulas

$$f(n) = n^2, \quad g(n) = 2^n.$$

Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, $g \circ f$.

8. Given $f = \{(a, b), (b, a), (c, b)\}$, a function from $X = \{a, b, c\}$ to X .

(a) Write $f \circ f$ and $f \circ f \circ f$ as sets of ordered pairs.

(b) Define

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

Write f^{450} as a set of ordered pairs.

9. If $f \circ g$ is one-to-one, does it follow that f is one-to-one?

10. Show that the function $f: \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{2\}$ defined by $f(x) = \frac{2x-3}{x-3}$ is a bijection, and find the inverse function.

(*Hint: Pay attention to the domain and codomain.*)