MACM 101 — Discrete Mathematics I

Exercises on Induction. Complete by: Thursday, July 23rd at 11:59pm

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1. Prove that the following equality is true for all integers $n \geq 1$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

2. Verify that the following inequality is true for all integers $n \geq 1$

$$\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}.$$

3. Use induction to prove that $11^n - 6$ is divisible by 5 for all integers $n \geq 1$.

4. Let the numbers $c_1, c_2, \ldots$ be defined inductively as follows: $c_1 = 0$ and for every $n > 1$ the number c_n is given by $c_n = 4c_{\lfloor n/2 \rfloor} + n$.

   Prove that $c_n \leq 4(n-1)^2$ for all integers $n \geq 1$.

5. Let the numbers $e_0, e_1, e_2, \ldots$ be defined inductively as follows:

$$\begin{aligned} &e_0 = 12, \qquad e_1 = 29 \\ &e_n = 5e_{n-1} - 6e_{n-2} \quad \text{for } n \geq 1. \end{aligned}$$

   Prove that $e_n = 5 \cdot 3^n + 7 \cdot 2^n$ for all integers $n \geq 1$.

6. The kth harmonic number is defined to be $H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$. Prove that for all $n \in \mathbb{N}$ harmonic numbers satisfy the equality

$$H_1 + H_2 + \cdots + H_n = (n+1)H_n - n$$

7. Fibonacci numbers $F_1, F_2, F_3, \ldots$ are defined by the rule: $F_1 = F_2 = 1$ and $F_k = F_{k-2} + F_{k-1}$ for $k > 2$.
Lucas numbers $L_1, L_2, L_3, \ldots$ are defined in a similar way by the rule: $L_1 = 1$, $L_2 = 3$ and $L_k = L_{k-2} + L_{k-1}$ for $k > 2$.
Show that Fibonacci and Lucas numbers satisfy the following equality for all $n \geq 2$
$$L_n = F_{n-1} + F_{n+1}.$$

8. The game of Chomp is played by two players. In this game, cookies are laid out on a rectangular grid. The cookie in the top left position is poisoned. The two players take turns making moves; at each move, a player is required to eat a remaining cookie, together with all cookies to the right and/or below (that is all the remaining cookies in the rectangle, in which the first cookie eaten is the top left corner). The loser is the player who has no choice but to eat the poisoned cookie. Prove that if the board is square (and bigger than 1×1) then the first player has a winning strategy.

9. Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute rs. Show that no matter how you split the piles, the sum of the products computed at each step equals $\frac{n(n-1)}{2}$.

10. A complete binary tree is a graph defined through the following recursive definition.
Basis step: A single vertex is a complete binary tree.
Inductive step: If T_1 and T_2 are disjoint complete binary trees with roots r_1, r_2, respectively, the the graph formed by starting with a root r, and adding an edge from r to each of the vertices r_1, r_2 is also a complete binary tree.
The set of leaves of a complete binary tree can also be defined recursively.
Basis step: The root r is a leaf of the complete binary tree with exactly one vertex r.
Inductive step: The set of leaves of the tree T built from trees T_1, T_2 is the union of the sets of leaves of T_1 and the set of leaves of T_2.

The height $h(T)$ of a binary tree is defined in the class.
Use structural induction to show that $\ell(T)$, the number of leaves of a complete binary tree T, satisfies the following inequality

$$\ell(T) \leq 2^{h(T)}.$$