# Propositional Logic

*Discrete Mathematics*

*Courtesy of Andrei Bulatov*

# What is Logic?

*"Computer science is a mere continuation of logic by other means"*

*Georg Gottlob*

*"Contrariwise", continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic"*

*Lewis Carroll, Through the Looking Glass*

# Use of Logic

- *In mathematics and rhetoric:*
  - *Give precise meaning to statements.*
  - *Distinguish between valid and invalid arguments.*
  - *Provide rules of `correct' reasoning.*

*Natural language can be very ambiguous*

*`If you do your homework, then you'll get to watch the game.'*

≈

*`If you don't do your homework, then you will not get to watch ...'*

*`Do your homework or you'll fail the exam.'*

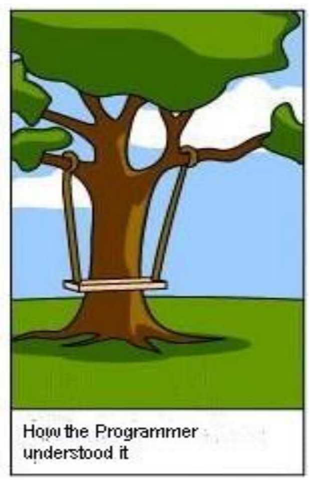
=

*`If you don't do your homework, then you'll fail the exam.'*

# Use of Logic (cntd)

- *In computing:*
  - *Derive new data / knowledge from existing facts*
  - *Design of computer circuits.*
  - *Construction of computer programs.*
  - *Verification of correctness of programs and circuit design.*
  - *Specification*


*What the customer really needed*




*How the Programmer understood it*


*What the customer got*

# Propositions (Statements)

- *Propositional logic deals with propositions (statements) and their truth values*
- *A proposition (statement) is a declarative sentence that is true or false*
- *Truth values are TRUTH (T or 1) and FALSE (F or 0).*
- *Examples:*
  - $1 + 1 = 2$ *(proposition, T)*
  - *The moon is made of cheese (proposition, F)*
  - *Go home! (not proposition, imperative)*
  - *What a beautiful garden! (not proposition, exclamation)*
  - $y + 1 = 2$ *(not proposition, uncertain)*
  - *God exists (proposition, ?)*

# Compound Propositions

- *Simplest propositions are called* *primitive propositions*

  *We shall use* *propositional variables* *to denote primitive propositions,* $p, q, r, \ldots$

- *We cannot decide the truth value of a primitive proposition. This is not what logic does.*

- *Instead we combine primitive propositions by means of logic connectives into* *compound propositions* *or* *formulas* *and look how the truth value of a compound proposition depends on the truth values of the primitive propositions it includes.*

  *We will denote compound propositions by* $\Phi, \Psi, \ldots$

# Logic Connectives

- *negation (not, $\neg$ )* *`It is not sunny'*
- *conjunction (and, $\wedge$ )* *`Today it is cloudy and raining'*
- *disjunction (or, $\vee$ )* *`In Winter it is raining or it is cloudy'*
- *implication (if…, then…, $\rightarrow$ )* *`If it is raining, then the ground is wet'*
- *exclusive or (either …, or …, $\oplus$ )* *`Either it is sunny or cloudy'*
- *biconditional (if and only if, $\leftrightarrow$ )* *`It is sunny if and only if it is summer'*

# Truth Tables

*Truth table is a way to specify the exact dependence of the truth value of a compound proposition through the values of primitive propositions involved*

*truth values of primitive propositions (propositional variables)*

*truth value of compound propositions (formulas)*

| $p$ | $q$ | $\Phi$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| … | … | … |
| | | |

# Truth Tables of Connectives (Negation and Conjunction)

- *Negation*

| $p$ | $\neg p$ |
|---|---|
| F (0) | T (1) |
| T (1) | F (0) |

*Unary connective*

*p: `Today is Friday'*

$\Rightarrow$

*$\neg$p: `Today is not Friday'*

- *Conjunction*

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

*Binary connective*

*p: `Today is Friday'*
*q: `It is raining'*

$\Rightarrow$

*$p \wedge q$: `Today is Friday and it is raining'*

# Truth Tables of Connectives (Disjunction)

- *Disjunction – inclusive `or'*

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

*p: `Students who have taken calculus can take this course'*
*q: `Students who have taken computing can take this course'*

$\Rightarrow$

*$p \vee q$: `Students who have taken calculus or computing can take this course'*

*Be careful with `or' constructions in natural languages!*

*`Do your homework, or you'll fail the exam.'*

*`Today is Friday or Saturday'*

# Truth Tables of Connectives (Exclusive or)

- *Exclusive `or'*

  *One of the propositions is true but not both*

| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

*`You can follow the rules or be disqualified.'*

*`Natalie will arrive today or Natalie will not arrive at all.'*

# Truth Tables of Connectives (Implication)

- *Implication*

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

*Note that logical (material) implication does not assume any causal connection.*

**`If** *black is white,* **then** *we live in the Antarctic.'*

**'If** *pigs fly,* **then** *Paris is in France.'*

# Implication as a Promise

- *Implication can be thought of as a promise, and it is true if the promise is kept*

  `**If** *I am elected,* **then** *I will lower taxes'*

  - *He is not elected and taxes are not lowered* — *promise kept!*
  - *He is not elected and taxes are lowered* — *promise kept!*
  - *He is elected, but (=and) taxes are not lowered* — *promise broken!*
  - *He is elected and taxes are lowered* — *promise kept!*

# Playing with Implication

- *Parts of implication*

$$p \rightarrow q$$

*hypothesis*
*antecedent*
*premise*

*conclusion*
*consequence*

`**if** *p*, **then** *q*’
`**if** *p*, *q*’
`*p* **is sufficient for** *q*’
`*q* **if** *p*’
`*q* **when** *p*’

`*q* **unless** $\neg$*p*’
`*p* **implies** *q*’
`*p* **only if** *q*’
`*q* **whenever** *p*’
`*q* **follows from** *p*’

`*a* **sufficient condition for** *q is p*’
`**a necessary condition** *for p is q*’

# Playing with Implication (cntd)

- *Converse, contrapositive, and inverse*

$p \rightarrow q$ *`The home team wins whenever it is raining'*
*(`If it is raining then the home team wins')*

- Converse $q \rightarrow p$

*`If the home team wins, then it is raining'*

- Contrapositive $\neg q \rightarrow \neg p$

*`If the home team does not win, then it is not raining'*

- Inverse $\neg p \rightarrow \neg q$

*`If it is not raining, then the home team does not win'*

# Truth Tables of Connectives (Biconditional)

- *Biconditional or Equivalence*

  *One of the propositions is true if and only if the other is true*

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

*`You can take the flight if and only if you buy a ticket.'*

# Example

*`You can access the Internet from campus if you are a computer science major or if you are not a freshman.'*

$p$ - *`you can access the Internet from campus'*

$q$ - *`you are a computer science major'*

$r$ - *`you are a freshman'*

# Practice

*Exercises from the Book:*

*7th edition: No. 1, 2, 13, 31(c), 32(b,d), 37(a) (page 13)*

*8th edition: No. 1, 2, 15, 33(c), 34(b,d), 39(a) (page 13)*