Propositional Logic II

Discrete Mathematics

Courtesy of Andrei Bulatov

Previous Lecture

- Propositions, primitive and compound
  - p where p is "it is not raining"
  - ¬p where p is "it is raining"

Same meaning

- Logic connectives:
  - negation ¬
  - conjunction ∧
  - disjunction ∨
  - exclusive or ⊕
  - implication →
  - biconditional ↔
- Truth tables

p	q	nand(p∧q)
0	0	1
0	1	1
1	0	1
1	1	0

Example: converting English to logic

`You can access the Internet from campus if you are a computer science major or if you are not a freshman.'

p - `you can access the Internet from campus'

q - `you are a computer science major'

r - `you are a freshman'

$(q \vee \neg r) \rightarrow p$

Nested formulas

- Can also combine compound propositions with logical connectives

$$\text{e.g. } (p \to q) \to p$$

- Truth value depends on the truth value of the sub-formulas

p	q	$p \to q$	$(p \to q) \to p$
0	0	1	0
0	1	1	0
1	0	0	1
1	1	1	1

An Example

- Construct the truth table of the following compound proposition

$$p \rightarrow (\neg q \vee p)$$

p	q	$\neg q \vee p$	$p \rightarrow (\neg q \vee p)$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

Tautologies

- A Tautology is a compound proposition (formula) that is true for all combinations of truth values of its propositional variables

$$(p \rightarrow q) \vee (q \rightarrow p)$$

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \vee (q \rightarrow p)$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	1	1	1

“To be or not to be”

Contradictions

- Contradiction is a compound proposition (formula) that is false for all combinations of truth values of its propositional variables

$(p \oplus q) \wedge (p \oplus \neg q)$

p	q	$p \oplus q$	$p \oplus \neg q$	$(p \oplus q) \wedge (p \oplus \neg q)$
0	0	0	1	0
0	1	1	0	0
1	0	1	0	0
1	1	0	1	0

"Black is white and black is not white"

Example

   

Write the following as a propositional formula and construct the truth table

> "An inhabitant of a castle in Transylvania is either sane or insane, and is a human or a vampire"

sane insane human vampire

$$(sane \oplus insane) \land (human \lor vampire)$$

s	i	h	v	$s \oplus i$	$h \lor v$	
0	0	0	0	0	0	0

Example

Write the following as a propositional formula and construct the truth table

“If a person is an insane vampire then he believes
only in false things and always lies”

Logic Equivalences

- Compound propositions Φ and Ψ are said to be logically equivalent if the proposition Φ is true (false) if and only if Ψ is true (respectively, false)

or

- The truth tables of Φ and Ψ are equal

or

- For any choice of truth values of the primitive propositions (propositional variables) of Φ and Ψ, formulas Φ and Ψ have the same truth value

- If Φ and Ψ are logically equivalent, we write

$$\Phi \Leftrightarrow \Psi$$

Why Logic Equivalences

- To simplify compound propositions

``If you are a computer science major or a freshman and you are not a computer science major or you are granted access to the Internet, then you are a freshman or have access to the Internet”

- To convert complicated compound propositions to certain `normal form’ that is easier to handle

Conjunctive Normal Form   CNF

Example Equivalences

- Implication and its contrapositive

p	q	$p \to q$	$\neg q \to \neg p$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

- All tautologies are equivalent to T

- All contradictions are equivalent to F

Equivalences and Tautologies

- **Theorem** Compound propositions Φ and Ψ are logically equivalent if and only if $\Phi \leftrightarrow \Psi$ is a tautology.

To **prove** the above theorem, need to show two things:

- "if" direction:
  if $\Phi \leftrightarrow \Psi$ is a tautology, then $\Phi \Leftrightarrow \Psi$
- "only if" direction:
  if $\Phi \Leftrightarrow \Psi$, then $\Phi \leftrightarrow \Psi$ is a tautology

Equivalences and Tautologies cont.

- **Theorem** Compound propositions Φ and Ψ are logically equivalent if and only if $\Phi \leftrightarrow \Psi$ is a tautology.

Proof ("if" direction):

- Assume $\Phi \leftrightarrow \Psi$ is a tautology.
- Then by definition of a tautology, $\Phi \leftrightarrow \Psi$ is True for every choice of truth values for primitive propositions.
- Since $\Phi \leftrightarrow \Psi$ is True if and only if Φ and Ψ have the same truth value, for every choice of truth values of propositional variables, Φ and Ψ have the same truth value.
- Hence by definition, $\Phi \Leftrightarrow \Psi$

Equivalences and Tautologies cont.

- **Theorem** Compound propositions Φ and Ψ are logically equivalent if and only if $\Phi \leftrightarrow \Psi$ is a tautology.

Proof ("only if" direction):

- Assume $\Phi \Leftrightarrow \Psi$ ~~is a tautology.~~
- Then by definition of $\Phi \Leftrightarrow \Psi$, for every choice of truth values of propositional variables, Φ is True if and only if Ψ is True.
- Then $\Phi \leftrightarrow \Psi$ is True for every choice of truth values of propositional variables.
- Hence by definition, $\Phi \leftrightarrow \Psi$ is a Tautology.

Q.E.D.

Practice

Exercises from the Book:

7th edition: No. 10(c,d), 14, 16, 17, 22, 31 (page 35)

8th edition: No. 12(c,d), 18, 20, 21, 26, 35 (page 38)

Practice

- Construct a truth table

$$(p \to q) \vee (\neg p \to r)$$

- Are these system specifications consistent (not a contradiction)?

  - The router can send packets to the edge system only if it supports the new address space

  - For the router to support the new address space it is necessary that the latest software release be installed.

  - The router can send packets to the edge system if the latest software release is installed.

  - The router does not support the new address space.