Logical Inference

Discrete Mathematics

Courtesy of Andrei Bulatov

Previous Lecture

- *Valid and invalid arguments*
- *Logic inference*
- *Rules of inference*
  - *Modus Ponens*
  - *Rule of Syllogism*
  - *Modus Tollens*
  - *Rule of Disjunctive Syllogism*
  - *Rule for Proof by Cases*
  - *Rule of Contradiction*
  - *Rule of Simplification*
  - *Rule of Addition*

Logical Inference

- *The goal of an argument is to infer the required conclusion from given premises*

- *A argument is a sequence of statements, each of which is either*
  - *(1) a premise, or*
  - *(2) obtained from preceding statements by a rule of inference*

Example

- *Premises:*

``It is not sunny this afternoon and it is colder than yesterday.
We will go swimming only if it is sunny.
If we do not go swimming, then we will take a canoe trip.
If we take a canoe trip, then we will be home by sunset.''

- *Conclusion: ``We will be home by sunset.''*
- *Notation:*

  p - it is sunny this afternoon
  q - it is colder than yesterday
  r - we will go swimming
  s - we will take a canoe trip
  t - we will be home by sunset

- *Premises:* $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, *and* $s \rightarrow t$
- *Conclusion:* t

Example (cntd)

- *We have* $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, *and* $s \rightarrow t$

Step	*Reason*
1. $\neg p \wedge q$	*premise*
2. $\neg p$	*simplification (1)*
3. $r \rightarrow p$	*premise*
4. $\neg r$	*modus tollens (2,3)*
5. $\neg r \rightarrow s$	*premise*
6. s	*modus ponens (4,5)*
7. $s \rightarrow t$	*premise*
8. t	*modus ponens (6,7)*

Logic Puzzles

- *A prisoner must choose between two rooms each of which contains either a lady, or a tiger. If he chooses a room with a lady, he marries her, if he chooses a room with a tiger, he gets eaten by the tiger. The rooms have signs on them:*

I	II
at least one of these rooms contains a lady	*a tiger is in the other room*

It is known that either both signs are true or both are false

Logic Puzzles (cntd)

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- *Notation:*

  p - *the first room contains a lady*

  q - *the second room contains a lady*

- *Premises:*

  $(p \vee q) \leftrightarrow \neg p$

  $(p \vee q) \rightarrow \neg p,$

  $\neg p \rightarrow (p \vee q)$

Logic Puzzles (cntd)

- *Argument*

$(p \vee q) \rightarrow \neg p,$
$\neg p \rightarrow (p \vee q)$

Step	*Reason*
1. $\neg p \rightarrow (p \vee q)$	*premise*
2. $p \vee (p \vee q)$	*expression for implication (1)*
3. $p \vee q$	*laws of associativity & idempotence (2)*
4. $(p \vee q) \rightarrow \neg p$	*premise*
5. $\neg p$	*modus ponens (3,4)*
6. q	*rule of disjunctive syllogism (3,5)*

Conjunctive Normal Form

- *A* ***literal*** *is a primitive statement (propositional variable) or its negation*

  $p, \ \neg p, \ q, \ \neg q$

- *A* ***clause*** *is a disjunction of one or more literals*

  $p \vee q, \ p \vee \neg q \vee r, \ \neg q, \ \neg s \vee s \vee \neg r \vee \neg q$

- *A statement is said to be a Conjunctive Normal Form (CNF) if it is a conjunction of clauses*

  $p \wedge (p \vee \neg q) \wedge (\neg r \vee \neg p)$

  $p \wedge q \wedge (\neg r \vee \neg p)$

  $(\neg r \vee q) \wedge (p \vee \neg q \vee \neg s \vee r) \wedge (\neg r \vee \neg p)$

  $\neg r$

CNF Theorem

$p \vee (q \wedge r) \Rightarrow (p \vee q) \wedge (p \vee r)$

- **Theorem**

Every statement is logically equivalent to a formula in CNF form.

Informally,

Let Φ *be a (compound) statement.*

Step 1. Express all logic connectives in Φ *through negation, conjunction, and disjunction. Let* Ψ *be the obtained statement.*

Step 2. Using DeMorgan's laws move all the negations in Ψ *to individual primitive statements. Let* Θ *denote the updated statement*

Step 3. Using distributive laws transform Θ *into a CNF.*

Example

Find a CNF logically equivalent to $(p \rightarrow q) \rightarrow r$

$$\neg(p \rightarrow q) \vee r$$

$$\Leftrightarrow \neg(\neg p \vee q) \vee r$$

$$\Leftrightarrow (\neg\neg p \wedge \neg q) \vee r$$

$$\Leftrightarrow (p \wedge \neg q) \vee r$$

$$\Leftrightarrow (p \vee r) \wedge (\neg q \vee r)$$

Rule of Resolution

- $p \vee q$
  $\neg p \vee r$
  $\therefore\ q \vee r$

  $q \vee r$ is called the ***resolvent***

- *The corresponding tautology* $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

``Jasmine is skiing or it is not snowing.
It is snowing or Bart is playing hockey."

p - `it is snowing'

q - `Jasmine is skiing'

r - `Bart is playing hockey'

``Therefore, Jasmine is skiing or Bart is playing hockey"

Resolution and rules of inference

- *Resolution subsumes most rules of inference*

Modus ponens

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array} \quad \rightarrow \quad \begin{array}{l} \neg p \vee q \\ p \\ \hline q \end{array}$$

Modus tollens

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array} \quad \rightarrow \quad \begin{array}{l} \neg p \vee q \\ \neg q \\ \hline \neg p \end{array}$$

Syllogism

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array} \quad \rightarrow \quad \begin{array}{l} \neg p \vee q \\ \neg q \vee r \\ \hline \neg p \vee r \end{array}$$

Contradiction

$$\begin{array}{l} p \to F \\ \hline \therefore \neg p \end{array}$$

Method of resolution

- *Together with CNF form, resolution provides a powerful, algorithmic means of proving inferences*
- *To prove an inference* $(p_1 \land p_2 \land p_3 \land \cdots \land p_n) \rightarrow q$ *is valid, instead prove that*

$$p_1 \land p_2 \land p_3 \land \cdots \land p_n \land \neg q$$

is a contradiction

- *Write the above in CNF form & use resolution to infer the empty clause (∅). That is, infer the contradiction*

$$\begin{array}{l} p \\ \neg p \\ \hline \therefore \varnothing \end{array}$$

Example (a lady and a tiger)

- *Prove* $((p \vee q) \leftrightarrow \neg p) \rightarrow q$ *is a tautology (valid inference)*
- *Resolution: prove that* $((p \vee q) \leftrightarrow \neg p) \wedge \neg q$ *is a contradiction*
- *1st step: Write clauses in CNF form*

1. $(p \vee q) \rightarrow \neg p$
2. $\neg p \rightarrow (p \vee q)$
3. $\neg q$

Example (a lady and a tiger, cont.)

- *Clauses:* $\neg p,\ p \vee q,\ \neg q$
- *Argument:*

Step	*Reason*
1. $\neg p$	*premise*
2. $p \vee q$	*premise*
3. q	*Resolution (1,2)*
4. $\neg q$	*premise*
5. $\varnothing$	*Resolution (3,4)*

Practice

Exercises from the Book:

7^{th} edition: No. 11, 12, 33 (page 79 - 80)

8^{th} edition: No. 11, 12, 33 (page 83 – 84)

- *Prove that resolution is a valid rule of inference*
- *Same arrangements as in the `A lady or a tiger' problem. This time if a lady is in Room I, then the sign on it is true, but if a tiger is in it, then the sign is false. If a lady is in Room II, then the sign on it is false, and if a tiger is in it, then the sign is true. Signs are*

I	II
both rooms contain ladies	*both rooms contain ladies*