# Predicates and Quantifiers

*Discrete Mathematics*

*Courtesy of Andrei Bulatov*

# What Propositional Logic Cannot Do

- *We saw that some declarative sentences are not propositions without specifying the value of `indeterminates'*

  *``x + 2 is an even number"*

  *``If $x + 1 > 0$, then $x > 0$"*

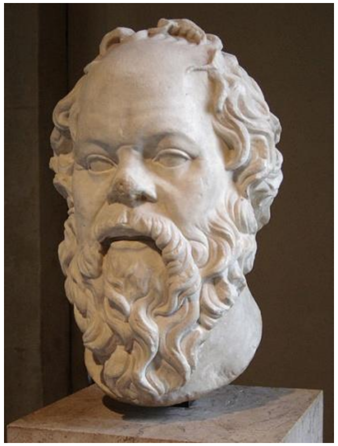
  *``x drives a lambo"*

- *Some valid arguments cannot be expressed with all our machinery of tautologies, equivalences, and rules of inference*

  *Every man is mortal.*
  *Socrates is a man.*

  ---

  $\therefore$ *Socrates is mortal*



# Predicates

- *Sentences like `x is greater than 3' or `person x has a brother' are not true or false unless the variable is assigned some particular value.*
- *The sentence `x is greater than 3' consists of 2 parts.*
  - *The first part, x, is called the variable or the subject of the sentence.*
  - *The second part – the predicate `is greater than 3' – refers to a property the subject can have.*
- *Sentences that have such structure are called predicates*
- *We write P(x) to denote a predicate with variable x*

# Unary, Binary, and so on

*`x is greater than 3'*
*`x is my brother'*
*`x is a human being'*

*unary predicates contain only 1 variable*

$P(x)$

*`x is greater than y'*
*`x is the mother of y'*
*`car x has colour y'*

*binary predicates contain 2 variables*

$Q(x,y)$

*`x divides y + z'*
*`x sits between y and z'*
*`x is a son of y and z'*

*ternary predicates contain 3 variables*

$R(x,y,z)$

# Assigning a Value

- *When a variable is assigned a value, the predicate turns into a proposition, whose truth value can be evaluated.*

*P(x) = `x is greater than 3'*

- *x=2* → *P(2) = `2 is greater than 3'* *false*
- *x=4* → *P(4) = `4 is greater than 3'* *true*

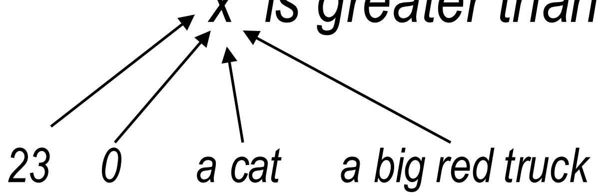
*Q(x,y) = `car x has colour y'*

- *x=my car, y=red* → *Q(my car,red) = `my car is red'* *true*
- *x=my car, y=grey* → *Q(my car,grey) = `my car is grey'* *false*

# Domain

- *We cannot assign a variable of a predicate ANY value. We need to obtain a meaningful statement!*

*`x is greater than 3'*




- *Every variable of a predicate is associated with a* ***domain*** *or* ***domain of discourse****, and its values are taken from this domain*

| | | |
|---|---|---|
| *`x is greater than 3'* | *x is a number* | |
| *`x is my brother'* | *x is a human* | |
| *`x is an animal'* | *x is a ???* | |
| *`car x has colour y'* | *x is a car*<br>*y is a colour* | *In CS lingo: variables are **typed*** |

# Example

- *R(x,y,z) = "x + y = z"*

  *What is the truth value of:*

  - *R(1,2,3)* True
  - *R(10,10,10)* False
  - *R(monkey, 2, 3)* undefined
  - *R(x, y, x + y)* True (kind of)

  for any x, y, R(x, y, x+y) is true

# Quantification

- *Consider the statement "for any numbers x, y, R(x,y,x+y) is true"*
- *This is a (universally) quantified statement*
- *Quantifiers are ways of obtaining propositions from predicates without giving specific values*
- *Examples:*

| | |
|---|---|
| *`Every man is mortal'*<br>*`There is x such that x is greater than 3'*<br>*`There is a person who is my father'*<br>*`For any* $x$, $x^2 \geq 0$ *'* | *quantification* |

# Universal Quantifiers

- *Abbreviates constructions like*

  *For all …*
  *For any …*
  *Every …*
  *Each …*

- *Asserts that a predicate is true for all values in the domain*

  *`Every man is mortal'*
  *`All lions are lazy'*
  *`For any* $x$*,* $x^2 \geq 0$ *'*

- *Notation:* $\forall$ forall

- $\forall x \; P(x)$ *is true if and only if P(a) is true for every a in the domain*

# Universal Quantifiers (cntd)

*`For any $x$, $x^2 \geq 0$ '* *true!*

*`Every car is grey'* *false! my car is not grey*

- $\forall x$ *P(x) is false if and only if* *there is* *at least one value a from the universe such that P(a) is false*
- *Such a value a is called a* *counterexample*

# Existential Quantifiers

- *Abbreviates constructions like*

    *For some …*
    *For at least one …*

    *There is …*
    *There exists …*

- *Asserts that a predicate is true for at least one value in the domain*

    *`There is a living king'*
    *`Some people are bad'*
    *`There is* $x$ *such that* $x^2 \geq 10$*'*

- *Notation:* $\exists$ exists

- $\exists x\ P(x)$ *is true if there is some value a in the domain such that P(a) is true*

# Existential Quantifiers (cntd)

*`There is a grey car'* — *true! I've seen them*

*`For some* $x$, $x^2 < 0$*'* — *false!*

- $\exists x$ *P(x) is false if and only if for all a from the universe P(a) is false*
- *Disproving an existential statement is difficult!*

# Quantifiers and Negations

- *Summarizing*

| | *true* | *false* |
|---|---|---|
| $\forall x\, P(x)$ | *For every value a from the universe P(a) is true* | *There is a counterexample – a value a from the universe such that P(a) is false* |
| $\exists x\, P(x)$ | *There is a value a from the universe such that P(a) is true* | *For all values a from the universe P(a) is false* |

- *Observe that*

$\neg\forall x\, P(x)$ *is true if and only if* $\exists x\, \neg P(x)$ *is true*

$\neg\exists x\, P(x)$ *is true if and only if* $\forall x\, \neg P(x)$ *is true*

DeMorgan laws

# Example

$\forall x\, P(x)$ $\exists x\, P(x)$

- *What is the negation of each of the following statements?*

| *Statement* | *Negation* |
|---|---|
| *All lions are lazy* | *There is an active lion* |
| *Everyone has two legs* | *There is a person who does not have two legs* |
| *Some people like coffee* | *All people hate coffee* |
| *There is a lady in one of these rooms* | *There is a tiger in every room* |

# Multiple quantifiers

- *Often predicates have more than one variable. In this case we need more than one quantifier.*

$P(x,y)$ = *``car x has colour y"*

| | |
|---|---|
| $\forall x \forall y \; P(x,y)$ | *``every car is painted all colours* |
| $\exists x \exists y \; P(x,y)$ | *``there is a car that is painted some colour* |
| $\forall x \exists y \; P(x,y)$ | *``every car is painted some colour* |
| $\exists x \forall y \; P(x,y)$ | *``there is a car that is painted all colours* |

# Partial quantification

- *A statement containing multiple variables may be quantified in only some variables.*

$P(x,y)$ = *``car x has colour y"*

$\forall y\ P(x,y)$ *``car x is painted all colours*

$\exists y\ P(x,y)$ *``car x is painted some colour*

*Variable y is bound by the quantifier*

*Variable x is free (not bound by a quantifier)*

# Free and bound variables

- *Quantifiers bind tightly, and the part of the formula it applies to is called its scope*

$$\forall y\ (......)\ ......$$

- *Occurrences of the quantified variable inside the scope are bound by the quantifier, and are free outside the scope*

$$\forall y\ Q(x,y) \lor \forall y(x < y) \lor R(y)$$

*bound* *bound* *free*

# Defining New Predicates

- *Partial quantification gives a new predicate*

  *``car x has some colour"* $Q(x) = \exists y\ P(x,y)$

- *Let P(x,y) mean ``lion x likes y"*

  $Q(y) = \forall x\ P(x,y)$ *``every lion likes y"* *Q(meat) is true*, *Q(apples) is false*

  $R(x) = \forall y\ P(x,y)$ *``lion x likes everything"* *R(x) is always false*

  *(say it using quantifiers:* $\forall x \neg R(x)$ *or* $\forall x \neg(\forall y\ P(x,y))$ *)*

  $S(x) = \exists y\ P(x,y)$ *``lion x likes some food"* *S(Bob) is true*, *S(Gob) is true*

  *May not be the same y for Bob and Gob!*

# Practice

*Exercises from the Book:*

*7th edition: No. 1, 5, 7, 10, 20 (page 53 – 54)*

*8th edition: No. 1, 5, 7, 10, 20 (page 56 – 57)*