Predicates and Quantifiers II

Discrete Mathematics

Courtesy of Andrei Bulatov

Previous Lecture

- *Predicates*
- *Assigning values, universe, truth values*
- *Quantifiers*
- *Quantifiers and negation*
- *Multiple quantifiers*

Quantifiers & predicates in logical formulas

- *We can combine predicates, quantifiers, and logical connectives to make more complicated statements*

``Every car is blue, or there is a red car"

P(x) - ``car x is blue"* *Q(x) - ``car x is red"

$(\forall x\, P(x)) \vee (\exists x\, Q(x))$

``For every number there is a smaller one, or there is the least number"

$x \leq y$ - "x is less than or equal to y"

$(\forall x\, \exists y\, (y \leq x)) \vee (\exists x\, \forall y\, (x \leq y))$

Quantifiers & predicates in logical formulas (cont.)

- *We can also quantify over compound formulas*

  ``*Every car is red or blue*”

  P(x) - ``car x is red”* *Q(x) - ``car x is blue”

  $\forall x\,(P(x) \vee Q(x))$

  ``*Everyone who has a current password can log onto the network*”

  P(x) - ``x has a current password”

  Q(x) - ``x can logon onto the network”

  $\forall x\,(P(x) \rightarrow Q(x))$

Scope of bound variables cont.

- *Consider the following, where*

  P(x) - ``car x is red"* *Q(x) - ``car x is blue"

$\forall x\,(P(x) \vee Q(x))$	*``every car is red or blue"*
$\forall x\,P(x) \vee Q(x)$	*``every car is red or car x is blue"*
$\forall x\,P(x) \vee \forall x\,Q(x)$	*``every car is red or every car is blue"*
$\forall x\,(P(x) \vee \forall x\,Q(x))$	*``For every car x, either x is red or all cars are blue"*

Example: (logically) defining new predicates

- *Can use bi-implication to logically define a predicate which is a compound formula over other predicates*

P(x,y) - ``y is a parent of x"

Q(x) - ``x is female"

S(x,y) - ``x and y are sisters"

$$S(x,y) \leftrightarrow Q(x) \wedge Q(y) \wedge \exists z\,(P(x,z) \wedge P(y,z))$$

Example: Mathematical definitions

Definition (limit):

A number A *is a limit of a sequence* $\{a_n\}$ *if for any number* $\varepsilon > 0$ *there is* N *such that for any* $n > N$ *we have* $|a_n - A| < \varepsilon$

$$\lim(\{a_n\}, A) \leftrightarrow \forall \varepsilon \, \exists N \, \forall n \; ((\varepsilon > 0) \wedge (n > N) \rightarrow (|a_n - A| < \varepsilon))$$

Example: Rules

- *Predicates and quantifiers are (implicitly) present in rules and laws*

  ``*If you have income more than $20000 you must file a tax report*”

  P(x) - ``x has income more than $20000”

  Q(x) - ``x must file a tax report”

  $\forall x\,(P(x) \rightarrow Q(x))$

Example: Theorems

- *Every theorem involves predicates and quantifiers*

``For every statement there is an equivalent CNF"

C(x) - ``x is a CNF" $\forall x \exists y (C(y) \wedge (x \Leftrightarrow y))$

``A parallelogram is a rectangle if all its angles are equal"

R(x) - ``parallelogram x is a rectangle"

A(x) - ``all angles of x are equal"

$\forall x (A(x) \rightarrow R(x))$

Examples

- *Professor's Carlson class has 29 students of which:*
  - *3 physics majors are juniors,*
  - *2 electrical engineering majors are juniors,*
  - *4 math majors are juniors,*
  - *12 physics majors are seniors,*
  - *4 el engineering majors are seniors,*
  - *2 el engineering majors are grad students,*
  - *2 math majors are grad students.*

p – physics major
m – math major
e – el eng major
j – junior
s – senior
g - grad

Express symbolically and evaluate the truth value:

1) There is a math major in the class who is a junior

$\exists x (m(x) \wedge j(x))$ T

Examples

- *Professor's Carlson class has 29 students of which:*
  - *3 physics majors are juniors,*
  - *2 electrical engineering majors are juniors,*
  - *4 math majors are juniors,*
  - *12 physics majors are seniors,*
  - *4 el engineering majors are seniors,*
  - *2 el engineering majors are grad students,*
  - *2 math majors are grad students.*

p – physics major
m – math major
e – el eng major
j – junior
s – senior
g - grad

Express symbolically and evaluate the truth value:

2) There is a senior in the class who is not a math major

$\exists x (S(x) \wedge \neg m(x))$ T

Examples

- *Professor's Carlson class has 29 students of which:*
  - *3 physics majors are juniors,*
  - *2 electrical engineering majors are juniors,*
  - *4 math majors are juniors,*
  - *12 physics majors are seniors,*
  - *4 el engineering majors are seniors,*
  - *2 el engineering majors are grad students,*
  - *2 math majors are grad students.*

p – physics major
m – math major
e – el eng major
j – junior
s – senior
g - grad

Express symbolically and evaluate the truth value:

3) Every student in the class is majoring in math or physics

$\forall x(m(x) \vee p(x))$ F

Examples

- *Professor's Carlson class has 29 students of which:*
  - *3 physics majors are juniors,*
  - *2 electrical engineering majors are juniors,*
  - *4 math majors are juniors,*
  - *12 physics majors are seniors,*
  - *4 el engineering majors are seniors,*
  - *2 el engineering majors are grad students,*
  - *2 math majors are grad students.*

p – physics major
m – math major
e – el eng major
j – junior
s – senior
g - grad

Express symbolically and evaluate the truth value:

4) No grad student in the class is a physics major

$\forall x(\neg(g(x) \wedge p(x)))$ T

Examples

- *Professor's Carlson class has 29 students of which:*
  - *3 physics majors are juniors,*
  - *2 electrical engineering majors are juniors,*
  - *4 math majors are juniors,*
  - *12 physics majors are seniors,*
  - *4 el engineering majors are seniors,*
  - *2 el engineering majors are grad students,*
  - *2 math majors are grad students.*

p – physics major
m – math major
e – el eng major
j – junior
s – senior
g - grad

Express symbolically and evaluate the truth value:

5) Every senior in the class is majoring in either physics or el engineering

$\forall x(S(x) \rightarrow (p(x) \vee e(x)))$ T

Practice

Exercises from the Book:

7th edition: No. 34, 38, 45, 50 (page 55 – 56), No. 25, 26 (page 67)

8th edition: No. 34, 40, 47, 52 (page 58 – 59), No. 25, 26 (page 71)

Represent in symbolic form

a definition "Jaywalk means to cross a roadway, not being a lane, at any place which is not within a crosswalk and which is less one block from an intersection at which traffic control signals are in operation".

a rule "No driver of a vehicle shall drive such vehicle on, over, or across any fire hose laid on any street or private road, unless directed so to do by the person in charge of such hose or a police officer"