

Predicate logic

Previous Lecture

- *Free and bound variables*
- *Multiple quantifiers and logic connectives*
- *Definitions, rules, etc.*

Interpretation

- A logic statement is meaningless

It only makes sense if we specify the domains and particular meanings of predicates

$$\forall x P(x)$$

Interpretation

domain: animals
P(x): x has horns



domain: cars
P(x): x is red



domain: numbers
P(x): x is even



Logical Equivalence of Predicates

- *Recall that two compound propositions Φ and Ψ are logically equivalent ($\Phi \Leftrightarrow \Psi$) if and only if $\Phi \leftrightarrow \Psi$ is a tautology.*
- *In other words, propositional logical equivalence means "equivalent regardless of the truth of propositional variables"*
- *In predicate logic, there are two types of unknowns:*
 - *The predicates themselves*
 - *The indeterminates*
- *How can we define logical equivalence?*

Logical Equivalence of Quantified Propositions

● Definition:

Two formulas in predicate logic are said to be **logically equivalent** if they are equivalent **for any** interpretation of predicates and any domain for their variables.

● E.g. De Morgan's laws for quantifiers:

$$\neg(\forall x P(x)) \Leftrightarrow \exists x \neg P(x)$$

$$\neg(\exists x P(x)) \Leftrightarrow \forall x \neg P(x)$$

Proving the DeMorgan quantifier laws

- *Have to show equivalence for any interpretation of predicates*
- *Do so by assuming an arbitrary predicate (i.e. making no assumptions about its meaning or domain), then **generalizing** it to all predicates*
- *E.g. $\neg(\forall x P(x)) \Leftrightarrow \exists x \neg P(x)$*
 - *Assume P is an arbitrary predicate.*
 - *If $\neg(\forall x P(x))$, then it is not the case that every value a satisfies $P(a)$*
 - *Pick one such a . Now $\neg P(a)$ holds, so $\exists x \neg P(x)$ does too*
 - *If on the other hand, $\exists x \neg P(x)$, then there exists a such that $\neg P(a)$ holds*
 - *So, it is not the case that $P(a)$ is true for all a , i.e. $\neg(\forall x P(x))$ holds*
 - *Therefore, equivalence holds for any interpretation of P*

Rules of universal generalization & instantiation

- $P(a)$ where a is arbitrary

$$\therefore \forall x P(x)$$

Rule of universal generalization

- $Q(P)$ in previous argument is

$$\text{"}\neg(\forall x P(x)) \Leftrightarrow \exists x \neg P(x)\text{"}$$



- *Can also instantiate a universal with particular value*

$$\forall x P(x)$$

$$\therefore \underline{P(a)}$$

Rule of universal instantiation

Implicit assumptions

- *Have to be careful with these types of proofs! You may be making **implicit assumptions** about the interpretation*
- *What's wrong with this argument showing $\forall x P(x) \rightarrow \exists x P(x) \Leftrightarrow T$*
 - *Assume P is arbitrary*
 - *If $\forall x P(x)$, then for every value a in the domain, $P(a)$ holds*
 - *Pick one value a in the domain such that $P(a)$ holds*
 - *Then $\exists x P(x)$ holds*

Universal distribution

● $\forall x (P(x) \wedge Q(x)) \Leftrightarrow (\forall x P(x)) \wedge (\forall x Q(x))$ *Universal distribution*

● *Prove for all P & Q!*

● $\forall x (P(x) \vee Q(x))$ is *not* equivalent to $(\forall x P(x)) \vee (\forall x Q(x))$

● *Counterexample:*

Domain: integers

$P(x): x \geq 0$

$Q(x): x \leq 0$

$\forall x (P(x) \vee Q(x))$

T

$(\forall x P(x)) \vee (\forall x Q(x))$

F

Existentials

- Existentials distribute over \vee but not \wedge , i.e.

$\exists x (P(x) \wedge Q(x))$ is *not* logically equivalent to $(\exists x P(x)) \wedge (\exists x Q(x))$

$\exists x (P(x) \vee Q(x))$ *is* logically equivalent to $(\exists x P(x)) \vee (\exists x Q(x))$

- Existential rules of inference are also opposite

$$\frac{P(a)}{\therefore \exists x P(x)}$$

Existential generalization

$$\frac{\exists x P(x)}{\therefore P(a)} \text{ where } a \text{ can be arbitrary}$$

Existential instantiation

Example

- “Everyone who reads the textbook will get an A. There is a student who is reading the textbook. Therefore there is a student who will get an A”

<i>Step</i>	<i>Reason</i>
1. $\forall x (T(x) \rightarrow A(x))$	<i>premise</i>
2. $\exists x T(x)$	<i>premise</i>
3. <u>$T(c)$</u>	<i>Existential instantiation (2)</i>
4. <u>$T(c) \rightarrow A(c)$</u>	<i>Universal instantiation (1)</i>
5. $A(c)$	<i>Modus ponens (3,4)</i>
6. $\exists x A(x)$	<i>Existential generalization (5)</i>

Equivalences from substitution

- Let $\Phi \Leftrightarrow \Psi$ be propositional formulas and $\Phi(x)$, $\Psi(x)$ denote the compound propositions obtained from Φ and Ψ by replacing every propositional variable (p, q, r, \dots) with predicates ($P(x), Q(x), R(x), \dots$). Then

$$\forall x \Phi(x) \Leftrightarrow \forall x \Psi(x) \quad \text{and} \quad \exists x \Phi(x) \Leftrightarrow \exists x \Psi(x)$$

$$\begin{array}{ccc}
 p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r) & & \text{distributive law} \\
 \begin{array}{c} \color{red}{\downarrow} \\ \color{blue}{\downarrow} \\ \color{green}{\downarrow} \end{array} & \begin{array}{c} \color{red}{P(x)} \\ \color{blue}{Q(x)} \\ \color{green}{R(x)} \end{array} & \begin{array}{c} \color{red}{\downarrow} \\ \color{blue}{\downarrow} \\ \color{red}{\downarrow} \\ \color{green}{\downarrow} \end{array} \\
 \forall x (P(x) \wedge (Q(x) \vee R(x))) \Leftrightarrow \forall x ((P(x) \wedge Q(x)) \vee (P(x) \wedge R(x))) & &
 \end{array}$$

Examples

- $$\exists x \neg(P(x) \wedge Q(x)) \Leftrightarrow \exists x (\neg P(x) \vee \neg Q(x))$$

$$\underline{\neg(p \wedge q)} \Leftrightarrow \underline{\neg p \vee \neg q}$$

DeMorgan's law

- $$\exists x (P(x) \vee (Q(x) \vee R(x))) \Leftrightarrow \exists x ((P(x) \vee Q(x)) \vee R(x))$$

$$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$$

associativity

- $$\forall x (P(x) \vee (P(x) \wedge Q(x))) \Leftrightarrow \forall x P(x)$$

$$p \vee (p \wedge q) \Leftrightarrow p$$

absorption law

- $$\forall x (P(x) \vee \neg P(x)) \Leftrightarrow T$$

Permutation of Quantifiers

- Order of consecutive universals/existentials does not matter

$$\forall x \forall y P(x,y) \Leftrightarrow \forall y \forall x P(x,y)$$

$$\exists x \exists y P(x,y) \Leftrightarrow \exists y \exists x P(x,y)$$

- Order of alternating universals & existentials **does** matter

$\forall x \exists y P(x,y)$ and $\exists y \forall x P(x,y)$ are not equivalent

- Example:

$P(x,y) =$ “ y is the mother of x ”

$\forall x \exists y P(x,y) =$ “Everyone has a mother”

$\exists y \forall x P(x,y) =$

“There is a person who is the mother of everyone”



Practice

Exercises from the Book:

7th edition: No. 45, 50 (page 55 – 56)

8th edition: No. 47, 52 (page 58 – 59)

Represent in symbolic form

a definition “Jaywalk means to cross a roadway, not being a lane, at any place which is not within a crosswalk and which is less one block from an intersection at which traffic control signals are in operation”.

a rule “No driver of a vehicle shall drive such vehicle on, over, or across any fire hose laid on any street or private road, unless directed so to do by the person in charge of such hose or a police officer”