Sets

Discrete Mathematics

Courtesy of Andrei Bulatov

What is a Set

- *A set is an unordered collection of objects*

  --> This is not a rigorous definition!!!

- *A definition reduces a concept to other previously defined concepts*

  --> "A cow is a big animal with horns and four legs in the corners"

- *To create mathematical definitions, need some "initial concept" which forms a foundation for other definitions. This is usually a "set"*

- *Naïve set theory describes, informally, how sets are built and used*

- *Axiomatic set theory (ZFC) describes a formal, axiomatic system of reasoning about sets*

Elements, Describing a Set

- *The objects in a set are called* ***elements*** *or* ***members*** *of the set*

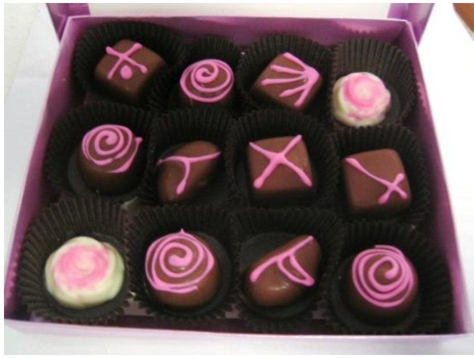
  $a \in A$ – *a is an element of A, a belongs to A*

  $a \notin A$ – *a is not an element of A, a does not belong to A*

- *One way to describe a set is to list its elements*

  $\{0,1,2,3,4,5,6,7,8,9\}$ – *the set of digits*

  $\{a,b,c,\ldots,x,y,z\}$ – *set of Latin letters, alphabet*



Extensionality: a set is defined by its elements

- *A set can be an element of another set*

  Set of alphabets: $\{\{a,b,c,\ldots\}, \{\alpha,\beta,\gamma,\ldots\},\ldots\}$

Paradoxes in set theory

- *We should be careful allowing sets to be contained in other sets*

- *Let U be the set of all sets that do not contain itself as an element:*

Strict barbers...?

- *Does U belong to itself?*

If yes, then $U \in U$, hence U does not satisfy the condition, and therefore $U \notin U$.

If not, then $U \notin U$, hence U does satisfy the condition, and therefore $U \in U$.

Universes and classes

- *To avoid paradoxes, separate sets and (proper) classes*
- *Classes are essentially sets --- don't worry about it too much*
- *A universe is the class which contains all objects we care about in a given context*
- *Examples:*

  Set = *the (proper) class of all sets*

  Prop = *the class of propositions*

  Num = *the class of numerical sets*

- *Modern foundations treat sets as concrete objects, and "classes" as "informal/naive sets"*

But we won't really care about that...

Building sets

$$\frac{-1}{1} \qquad \frac{1}{-1}$$

- *Sets can be described using* *set builder notation:*

  $\{x \mid P(x)\}$, *the set of all* x *such that* $P(x)$

  $\{x \mid$ *there is* y *such that* $x = 2y\}$, *the set of even numbers*

  $\parallel$

  $\{x \mid \exists y\ (x = 2y)\}$

  $\{x \mid x$ *is a black cow*$\}$

  $\mathbb{N} = \{0,1,2,3,\ldots\}$, *the set of natural numbers*

  $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,3,\ldots\}$, *the set of integers*

  $\mathbb{Q} = \{p/q \mid p,q$ *are integers and* $q \neq 0\}$, *the set of rationals*

  $\mathbb{Z}^+, \mathbb{Q}^+$, *the sets of positive integers and positive rationals*

  $\mathbb{R}$, *the set of real numbers*

Ambiguities in set builder

$\{ x \mid 1 \leq x \leq 10 \}$ <-- *What is this set?*

- *To avoid such ambiguities, can specify a set of objects from which x is drawn, i.e. a domain:*

$\underline{\{ x \in \mathbb{Z} \mid 1 \leq x \leq 10 \}}$ *the set of all integers from 1 to 10*

$= \{ x \mid x \in \mathbb{Z} \land 1 \leq x \leq 10 \}$

$\{ x \in \mathbb{Q} \mid 1 \leq x \leq 10 \}$ *the set of all rationals from 1 to 10*

$= \{ x \mid x \in \mathbb{Q} \land 1 \leq x \leq 10 \}$

Equality of Sets, Subsets

- *Two sets are* equal *if they have the same elements*

  That is, $A = B$ *if and only if* $\forall x\,(x \in A \leftrightarrow x \in B)$

$\{1,3,5\} = \{5,3,1\}$

$\{1,3,5\} = \{5,5,5,5,3,3,1,1,3,1\}$

$\{1,3,5\} \neq \{1,3,\{5\}\}$

- *Set* B *is a* subset *of a set* A *if every element of* B *is also an element of* A*.*

  That is $B \subseteq A$ *if and only if* $\forall x\,(x \in B \rightarrow x \in A)$

Example:

$\{ x \in \mathbb{Z} \mid 1 \leq x \leq 10 \}$ *describes a subset of* $\mathbb{Z}$

More definitions

- *Set B is superset of A, denoted $B \supseteq A$, if A is a subset of B*
- *Set B* is not *a subset, denoted $B \not\subseteq A$ of a set A if there exists an element in B but not A*

In symbols: $B \not\subseteq A \Leftrightarrow \exists x\,(x \in B \wedge x \notin A)$

- *A set B is said to be a* **proper subset** *of a set A, denoted by $B \subset A$, if it is a subset of A and is not equal to A.*

$B \subset A$

In symbols: $B \subset A \Leftrightarrow (B \subseteq A) \wedge \exists x\,(x \in A \wedge x \notin B)$

$\{1,3\} \subset \{1,3,5\}$

$\{1,3,5\} \supseteq \{1,3\}$

$\{2,6\} \not\subseteq \{1,4\}$

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[dr, phantom, "\subset"] & & \\
\mathbb{Z}^+ \arrow[ur, phantom, "\subset"] \arrow[dr, phantom, "\subset"] & & \mathbb{Q} \arrow[r, phantom, "\subset"] & \mathbb{R} \\
 & \mathbb{Q}^+ \arrow[ur, phantom, "\subset"] & &
\end{tikzcd}$$

Transitivity of subsets

- **Theorem**. *If* $A \subseteq B$ *and* $B \subseteq C$*, then* $A \subseteq C$.

  Proof.

  We have to prove that, for every x*, if* $x \in A$*, then* $x \in C$.

  Take any x *such that* $x \in A$*. Since* $A \subseteq B$*, this implies that* $x \in B$.

  Next, as $B \subseteq C$ *and* $x \in B$*, we have* $x \in C$.

  Q.E.D.

Homework: give a formal proof, using rules of inference

Equality and subsets

- **Theorem**. $A = B$ *if and only if* $A \subseteq B$ *and* $B \subseteq A$.

Proof.

If direction:

Suppose $A = B$.

Then for every x, if $x \in A$, then $x \in B$, so $A \subseteq B$

Likewise with $B \subseteq A$

Only if direction:

Suppose $A \subseteq B$ and $B \subseteq A$

Then if $x \in A$, then $x \in B$ by $A \subseteq B$

And if $x \in B$, then $x \in A$ by $B \subseteq A$, so $A = B$

Q.E.D.

Empty Set

- *Empty set is a set that has no elements.* $\emptyset$

  How many elements does this set contain? $\{\emptyset\}$ 1

- **Theorem.** *For any set A, (i)* $\emptyset \subseteq A$*, and (ii)* $A \subseteq A$.

  Proof.

  (i) We have to prove that $\forall x\,(\underline{x \in \emptyset} \rightarrow \underline{x \in A})$

  Since the premise $x \in \emptyset$ *is always false, we conclude that the implication* $x \in \emptyset \rightarrow x \in A$ *is always true.*

  Then we use the rule of universal generalization.

  (ii) Homework.

Cardinality

- *Let A be a set. If there are exactly n distinct elements in A where n is a natural number, then we say that A is a **finite set** and that n is the **cardinality** of A.*

  $$|A| = n$$

  $$|\{\{\emptyset\}\}| = 1$$

- *Examples*

  $|\emptyset| = 0$

  $|\{1,2,3,3\}| = 3$

  $|\{0,1,2,3,4,5,6,7,8,9\}| = 10$

  $|\{a,b,c,\ldots,x,y,z\}| = 26$

- *Sets that are not finite are called **infinite***

  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ *are infinite*

Practice

Exercises from the Book:

7th edition: 1, 5, 7, 9, 23 (page 125 – 126)

8th edition: 1, 7, 9, 11, 25 (page 131 – 132)