Operations on Sets

Discrete Mathematics

Courtesy of Andrei Bulatov

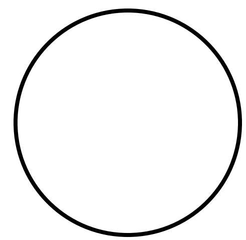
Previous Lecture

- *Sets, naive set theory*
- *Describing sets, set builder, Russell's paradox*
- *Subsets, equality of sets*
- *Empty sets*
- *Cardinality*

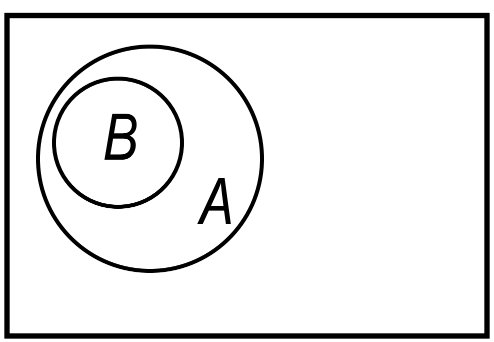
Venn Diagrams

- *Often it is convenient to visualize various relations between sets. We use* *Venn diagrams* *for that.*

universe



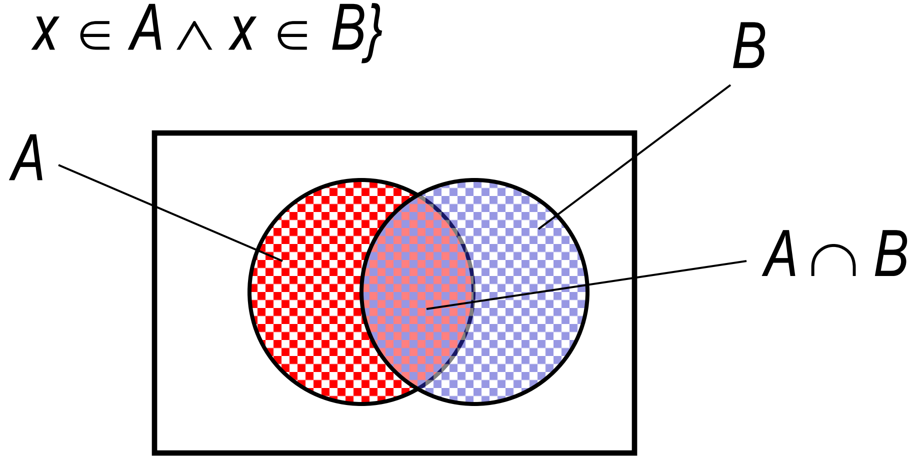
set



B is a subset of A

Intersection

- *The intersection of sets A and B, denoted by $A \cap B$, is the set that contains those elements that belong to both A and B.*
- $A \cap B = \{ x \mid x \in A \wedge x \in B \}$

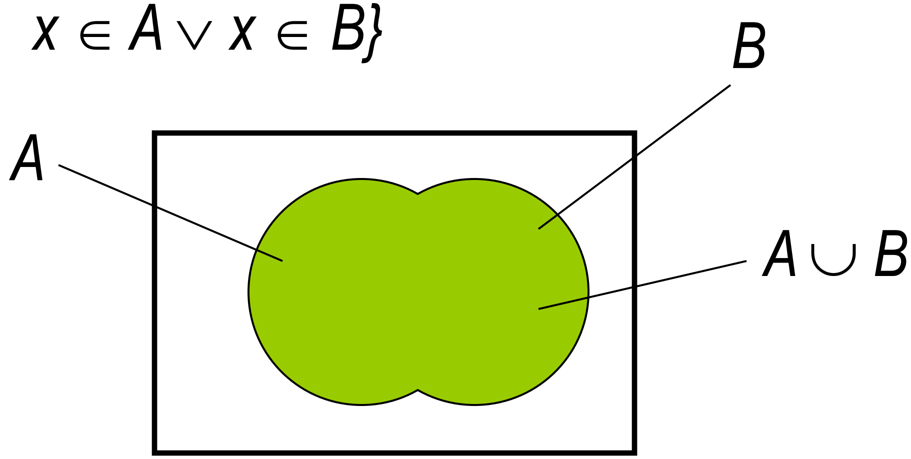


- *Examples*

$\{1,3,5,7\} \cap \{2,3,4,5,6\} = \{3,5\}$
$\{Jan.,Feb.,Dec.\} \cap \{Jan.,Feb.,Mar.\} = \{Jan.,Feb.\}$
$\{x \mid \exists y \ x=2y\} \cap \{x \mid \exists y \ x=3y\} = \{x \mid \exists y \ x=6y\}$
$\mathbb{Z} \cap \mathbb{Q}^+ = \mathbb{Z}^+$

Union

- *The* union *of sets* A *and* B*, denoted by* $A \cup B$*, is the set that contains those elements that are in* A *or in* B*.*
- $A \cup B = \{ x \mid x \in A \vee x \in B \}$



- *Examples*

$\{Mon, Tue, Wed, Thu, Fri\} \cup \{Sat, Sun\} = \{Mon, Tue, Wed, Thu, Fri, Sat, Sun\}$

$\{1,3,5,7\} \cup \{2,3,4,5,6\} = \{1,2,3,4,5,6,7\}$

Cardinality of the union

- *Given finite sets A and B, how many elements are in their union?*

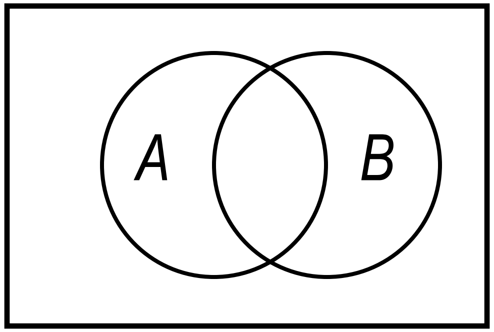
$$
\begin{aligned}
&|\{Mon,Tue,Wed,Thu,Fri\} \cup \{Sat,Sun\}| \\
&\qquad = |\{Mon,Tue,Wed,Thu,Fri,Sat,Sun\}| \\
&\qquad = |\{Mon,Tue,Wed,Thu,Fri\}| + |\{Sat,Sun\}|
\end{aligned}
$$

$$
\begin{aligned}
&|\{1,3,5,7\} \cup \{2,3,4,5,6\}| \\
&\qquad = |\{1,2,3,4,5,6,7\}| \\
&\qquad \neq |\{1,3,5,7\}| + |\{2,3,4,5,6\}|
\end{aligned}
$$

Disjoint Sets and Principle of Inclusion-Exclusion

- *Principle of inclusion-exclusion.* *For any finite sets* A *and* B

$$|A \cup B| = |A| + |B| - |A \cap B|$$



To count elements in $A \cup B$ *we first count elements of* A*, then elements of* B*. Elements of* $A \cap B$ *are counted twice, so, we subtract the number of such elements*

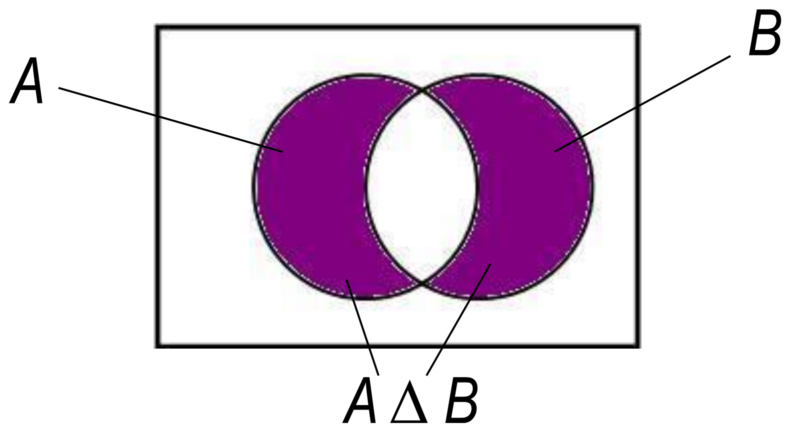
- *If* $|A \cap B| = 0$*, then* A *and* B *are said to be* *disjoint*

E.g. {Mon,Tue,Wed,Thu,Fri} and {Sat,Sun} are disjoint.

{1,3,5,7} and {2,3,4,5,6} are *not* *disjoint.*

Symmetric Difference

- *The symmetric difference of sets A and B, denoted by $A \Delta B$, is the set that contains those elements that are either in A or in B, but not in both.*
- $A \Delta B = \{ x \mid x \in A \oplus x \in B\}$



- *Example*

  {Jan.,Feb.,Mar.} Δ {Dec.,Jan.,Feb.} = {Dec.,Mar.}

Disjoint Sets and Symmetric Difference

- **Theorem**. *Sets A and B are disjoint if and only if*

$$A \cup B = A \Delta B$$

- *Proof.*

  $\Rightarrow$) *Suppose A and B are disjoint.*
  Notice first that $A \Delta B \subseteq A \cup B$. To prove the equality, it suffices to show that $A \cup B \subseteq A \Delta B$.
  Take $x \in A \cup B$. It belongs to A or B, but $x \notin A \cap B$, as the intersection is empty. Therefore, $x \in A \Delta B$.

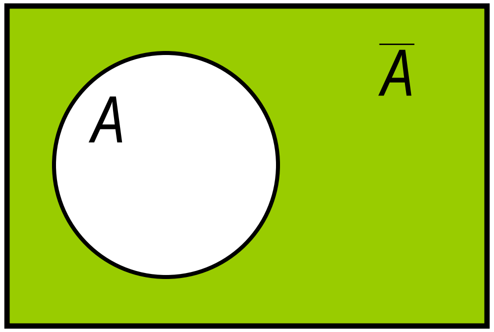
  $\Leftarrow$) *Prove by contrapositive. Assume $A \cap B \neq \varnothing$. Say $x \in A \cap B$*
  Then $x \in A \cup B$.
  However, from $x \in A \cap B$ we conclude that $x \notin A \Delta B$.
  Therefore, $A \cup B \neq A \Delta B$.

Q.E.D.

Complement

- *Let A be a set and U a universe, $A \subseteq U$. The* ***complement*** *of A, denoted by $\overline{A}$, is the set that comprises all elements of U that do not belong to A.*

$$\overline{A} = \{ x \mid x \in U \text{ and } x \notin A \} = \{ x \mid x \notin A \}$$



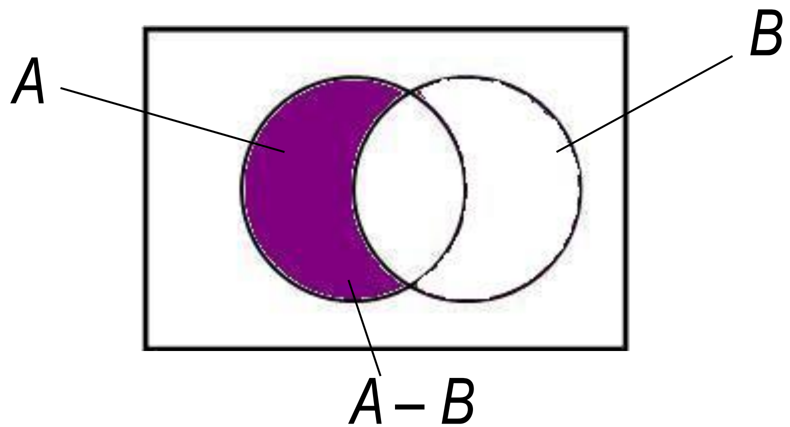
- *Let the universe be the set of all integers, and $A = \{ x \mid \exists y \ \ x=2y \}$*
  Then $\overline{A}$ is the set of all odd numbers

- *The universe is the Latin alphabet, $A = \{ a,e,i,o,u,y \}$.*
  Then $\overline{A} = \{ b,c,d,f,g,h,j,k,l,m,n,p,q,r,s,t,v,w,x,z \}$.

Difference

- *The difference of sets A and B (or relative complement of B in A), denoted by A – B, is the set containing those elements that are in A, but not in B.*

$$A - B = \{ x \mid x \in A \wedge x \notin B \}.$$



- $\{1,3,5\} - \{1,2,3\} = \{5\}$
- *Clearly,* $\overline{A} = U - A.$

Laws of Set Theory

- *Similar to logic connectives and formulas, expressions built from set operations and sets also satisfy some laws.*
- **Theorem**. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof. *We will show that* $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ *and* $\overline{A \cap B} \supseteq \overline{A} \cup \overline{B}$

Prove that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. *Take* $x \in \overline{A \cap B}$.

By the definition, $x \notin A \cap B$. *Therefore,* $x \notin A$ *or* $x \notin B$.

Hence $x \in \overline{A}$ *or* $x \in \overline{B}$. *Thus,* $x \in \overline{A} \cup \overline{B}$

Now we prove that $\overline{A \cap B} \supseteq \overline{A} \cup \overline{B}$. *Take* $x \in \overline{A} \cup \overline{B}$.

By definition, $x \in \overline{A}$ *or* $x \in \overline{B}$. *Therefore,* $x \notin A$ *or* $x \notin B$.

This implies $x \notin A \cap B$. *And, finally,* $x \in \overline{A \cap B}$.

Q.E.D.

Another Proof

- *Another way to prove equalities for sets is to use the set builder construction and some logic.*

$\overline{A \cap B} = \{ x \mid x \notin A \cap B \}$ *by definition of complement*

$= \{ x \mid \neg(x \in A \cap B) \}$ *by definition of does not belong symbol*

$= \{ x \mid \neg(x \in A \wedge x \in B) \}$ *by definition of intersection*

$= \{ x \mid \neg(x \in A) \vee \neg(x \in B) \}$ *by De Morgan's law*

$= \{ x \mid (x \notin A) \vee (x \notin B) \}$ *by def. of does not belong symbol*

$= \{ x \mid (x \in \overline{A}) \vee (x \in \overline{B}) \}$ *by definition of complement*

$= \{ x \mid x \in \overline{A} \cup \overline{B} \}$ *by definition of union*

$= \overline{A} \cup \overline{B}$

Q.E.D.

Sets and Logic

- *If we look closer at the second proof, we notice that there is a very tight connection between set operations and logic connectives. More on this next class...*

$\neg$ *corresponds to complement* $\overline{}$

$\vee$ *corresponds to union* $\cup$

$\wedge$ *corresponds to intersection* $\cap$

$\oplus$ *corresponds to symmetric difference* Δ

0 (false) corresponds to the empty set $\varnothing$

1 (truth) corresponds to the universe U

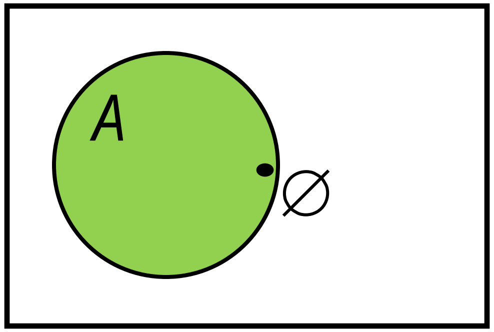
More Laws of Set Theory

$A \cup \varnothing = A$
$A \cap U = A$

Identity laws

$A \cup U = U$
$A \cap \varnothing = \varnothing$

Domination laws



$A \cup A = A$
$A \cap A = A$

Idempotent laws

$\overline{(\overline{A})} = A$

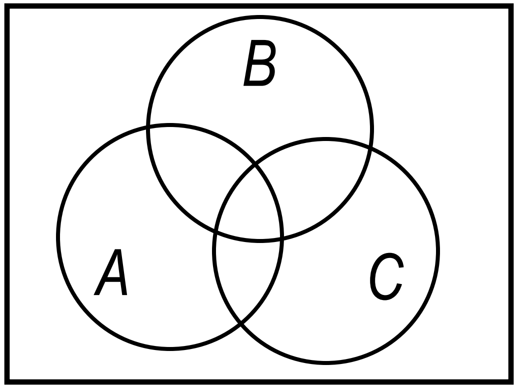
Complementation law

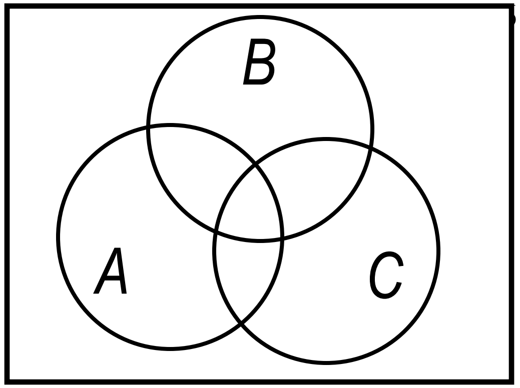
$A \cup B = B \cup A$
$A \cap B = B \cap A$

Commutative laws

$A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (B \cap C) = (A \cap B) \cap C$

Associative laws





More Laws of Set Theory (cntd)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Distributive laws

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

De Morgan's laws

$$A \cap (A \cup B) = A$$
$$A \cup (A \cap B) = A$$

Absorption laws

$$A \cup \overline{A} = U$$
$$A \cap \overline{A} = \varnothing$$

Complement laws

Practice

Exercises from the Book:

7th edition: 3, 16be, 19, 27, 42 (page 136 – 137)

8th edition: 3, 16be, 21, 29, 48 (page 144 – 145)