Sets and Logic

Discrete Mathematics

Courtesy of Andrei Bulatov

Previous Lecture

- *Venn diagrams*
- *Operations of*
  - *Intersection*
  - *Union*
  - *Symmetric difference*
  - *Complement*
  - *Difference*

- *De Morgan's law for sets* $\overline{A \cup B} = \bar{A} \cap \bar{B}, \quad \overline{A \cap B} = \bar{A} \cup \bar{B}$

Predicates and their meaning

- *Consider the predicates*

  $P(x,y) = x > y, \qquad Q(x,y) = y < x$

  Are they different?

- *Extensional view: a predicate is defined over a given domain by the set of values on which it is true (called its* **models***)*

"equivalent"

$$P(x) \equiv \{ x \mid P(x) \} = P_x$$

- *Now we can formally make a connection between logic and set operations*

$$P(x) \wedge Q(x) \equiv \{x \mid P(x) \wedge Q(x) \} = P_x \cap Q_x$$

Sets and Logic

- *Operation correspondence*

$$\neg P(x) \equiv \{ x \mid \neg P(x) \} = \overline{P_x}$$

$$P(x) \vee Q(x) \equiv \{ x \mid P(x) \vee Q(x) \} = P_x \cup Q_x$$

$$P(x) \oplus Q(x) \equiv \{ x \mid P(x) \oplus Q(x) \} = P_x \,\Delta\, Q_x$$

$$0 \equiv \emptyset$$

$$1 \equiv U$$

- *Law correspondence*

$$P(x) \Leftrightarrow Q(x) \equiv P_x = Q_x$$

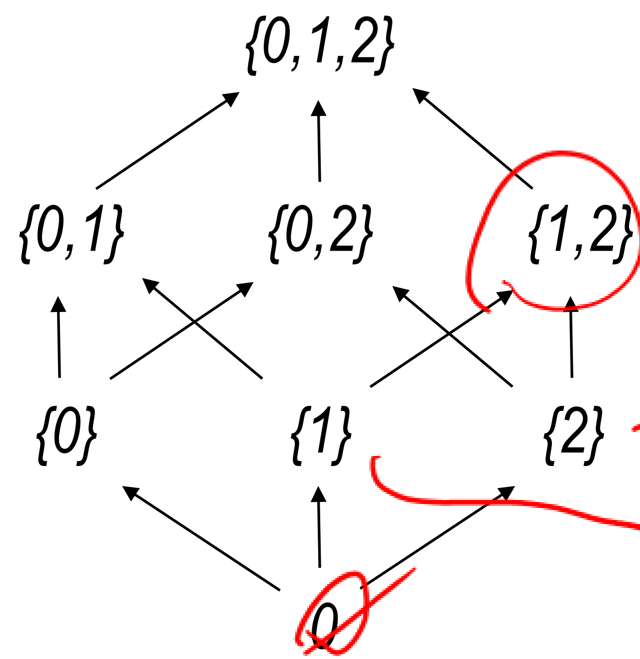
$$\neg(P(x) \vee Q(x)) \Leftrightarrow \neg P(x) \wedge \neg Q(x) \equiv \overline{P_x \cup Q_x} = \overline{P_x} \cap \overline{Q_x}$$

Power Set

- *Given a set A, the* power set *of A is the set of all subsets of A*

$$P(A) = \{ B \mid B \subseteq A \}$$

- *Example. Find* $P(\{0,1,2\})$ $= \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, A\}$



$= \{x \mid x = 1 \vee x = 2\}$

$\{x \mid x = 2\}$

$\{x \mid x = 1\}$

Propositional models

- *Given a set A of propositional variables, what might*
  - *A subset of A correspond to (in logic)?* Valuation (set of true propositions)
  - *The powerset P(A) correspond to?* Set of all valuations
  - *A proposition p correspond to (in sets)?* Set of valuations in which p is true
  - *A formula p ∧ q correspond to?* Set of valuations in which p ∧ q is true
- *Example: A = {p,q}*

	p	q	$p \wedge q$
{}	0	0	0
{q}	0	1	0
{p}	1	0	0
{p,q}	1	1	1

Rows = subsets of propositions

Cols = models of the formula

Cardinality of the power set

- *How many elements are in the powerset of a finite set?*

  ==> how many lines in a truth table over n variables?

- **Theorem**. *If A is a finite set, then* $|P(A)| = 2^{|A|}$

  Proof (sketch)

  Let B be a subset of A and |A|=n.

  We can describe B by which of the n elements of A it contains!

  Let $i_1 i_2 i_3 \ldots i_n$ *be the binary string where* $i_j = 1$ *if and only if B contains the jth element of A*

  How many unique strings (and hence subsets B) are there?

- *Representation of a subset of A where |A|=n as a binary string of length n is called an indicator vector or characteristic function*

Example

- Let $A = \{1,2,3,4,5,6,7,8,9,10\}$
- Write the following as indicator vectors $i_1 i_2 i_3 i_4 i_5 i_6 i_7 i_8 i_9 i_{10}$
  - $\{1,2,3\}$ → 1110000000
  - $\{1,3,5,7,9\}$ → 1010101010
  - The even digits → 0101010101
  - $\{\}$ 0000000000
  - $\{1,10\} \cup \{5\}$ 1000000001 $\{1,10\}$
    0000100000 $\{5\}$
  - $\{1,2\} \cap \{1,3\}$ 1000100001
  - $\overline{\{5\}}$ 1111011111