Relations

Discrete Mathematics

Courtesy of Andrei Bulatov

Previous Lecture

- *Set-based models of logic*
- *Power sets & propositional logic*
- *Binary representation of sets*

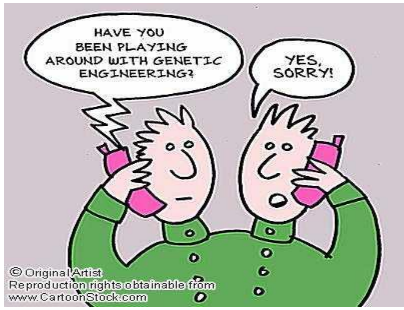
- *Fun fact:*
  there are $2^{2^{\wedge}n}$ *unique truth tables in n propositional variables*

- *Fun questions:*
  What does implication correspond to?
  What are the models of a quantified statement?

Relations

- *The predicate P(x,y) = x > y isn't a set of values in the same sense... it's a relationship that holds between the values of x and y*

- *Between people, familial relationships*



`to be brothers'	*x is a brother of y*
`to be older'	*x is older than y*
`to be parents'	*x and y are parents of z*

- *Between things, numerical relationships*

`to be greater than'	*x < y on the set of real numbers*
`to be divisible by'	*x is divisible by y on the set of integers*

- *Between things and people, legal relationships*

`to be an owner'	*x is an owner of y*

Relations as sets: the Cartesian Product

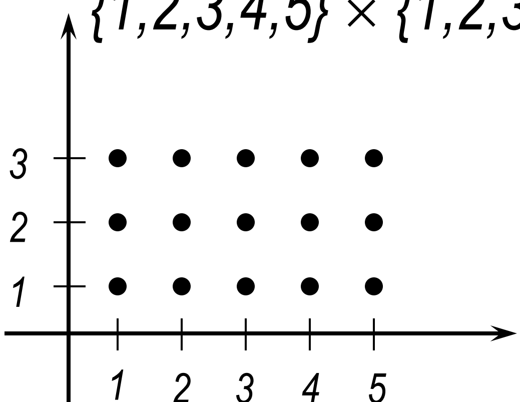
- *The* ***Cartesian product*** *of sets A and B, denoted by* $A \times B$*, is the set of all* ***ordered pairs*** *(a,b) of elements from A and B.*

  $$A \times B = \{ (a,b) \mid a \in A,\ b \in B \}$$

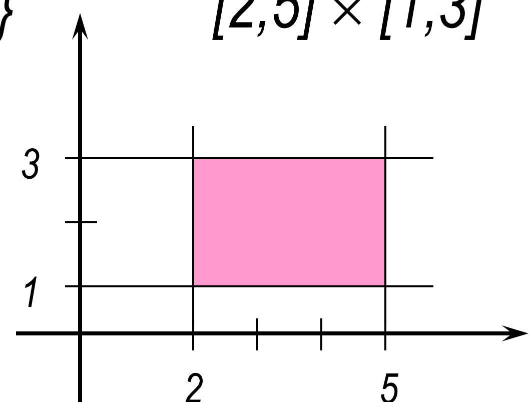
- *The elements of the Cartesian product are ordered pairs. In particular, (a,b) = (c,d) if and only if a = c and b = d.*

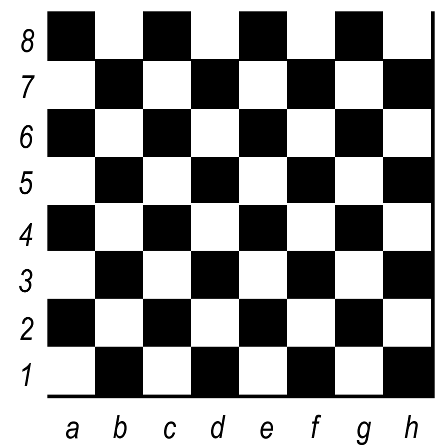
- *If sets are thought of as `1-dimensional' objects, Cartesian products are 2-dimensional*

$\{1,2,3,4,5\} \times \{1,2,3\}$



$[2,5] \times [1,3]$





Example

- *Are the following sets equal or not equal?*

  $\{1,2,3\}$ *and* $\{3,2,1\}$?

  $\{\{1,2,3\},\{3,2,1\}\}$ *and* $\{\{1,2,3\}\}$?

  $\{(1,2,3)\}$ *and* $\{(3,2,1)\}$?

  $\{(1,2,3), (3,2,1)\}$ *and* $\{(1,2,3)\}$?

Cardinality of Cartesian Product

- **Theorem.**

  $$| A \times B | = | A | \cdot | B |$$

- *Proof*

  Consider an element a of A

  We can construct |B| unique pairs (a,b) corresponding to the |B| elements of B

  Since we have |A| such unique elements a, in total there are $| A | \cdot | B |$ *unique ordered pairs.*

Q.E.D.

Back to predicates and their meaning

- *Consider the predicates*

  $P(x,y) = x > y,$ $\qquad\qquad$ $Q(x,y) = y > x$

- *They have 2 indeterminates, so they aren't quite sets of values in the same sense.*

- *They give a relationship between two indeterminates x and y, not just a set of values.*

- *Their models are hence sets of ordered pairs, i.e. a subset of the Cartesian product of the variable's domains called a relation*

$$P(x,y) \equiv \{ (x,y) \mid P(x,y) \} \subseteq U \times U$$

Binary Relations

- *A* ***binary relation*** *from set A to set B is any subset of $A \times B$.* *If $A = B$ then we say that the relation is* ***on*** *the set A*

`x is a brother of y'	$\subseteq$	People $\times$ People
`x is older than y'	$\subseteq$	People $\times$ People
`x is an owner of y'	$\subseteq$	People $\times$ Properties
`x < y'	$\subseteq$	$\mathbb{R} \times \mathbb{R}$
`x divides y'	$\subseteq$	$\mathbb{Z} \times \mathbb{Z}$

- ***Binary predicates*** *$P(x,y)$ correspond to binary relations!*

$$P(x,y) \equiv \{ (x,y) \mid P(x,y) \} \subseteq U \times U$$

Sets, Relations, and Predicates

- *Now binary predicates have a direct correspondence to sets too!*

Unary predicate	*Set*
$P(x)$	$A = \{ x \mid P(x) \}$
Binary predicate	*Binary relation*
$P(x,y)$	$R = \{ (x,y) \mid P(x,y) \}$
Ternary predicate	???
$P(x,y,z)$	

Cartesian Product of More Than Two Sets

- *Instead of ordered pairs we may consider ordered* ***triples****, or, more general,* ***k-tuples****.*

  (a,b,c), an ordered triple

  (a,b,c,d), an ordered quadruple

  $(a_1, a_2, \ldots, a_k)$ *a k-tuple*

- *Triples, quadruples, and k-tuples are elements of Cartesian products of 3, 4, and k sets, respectively*

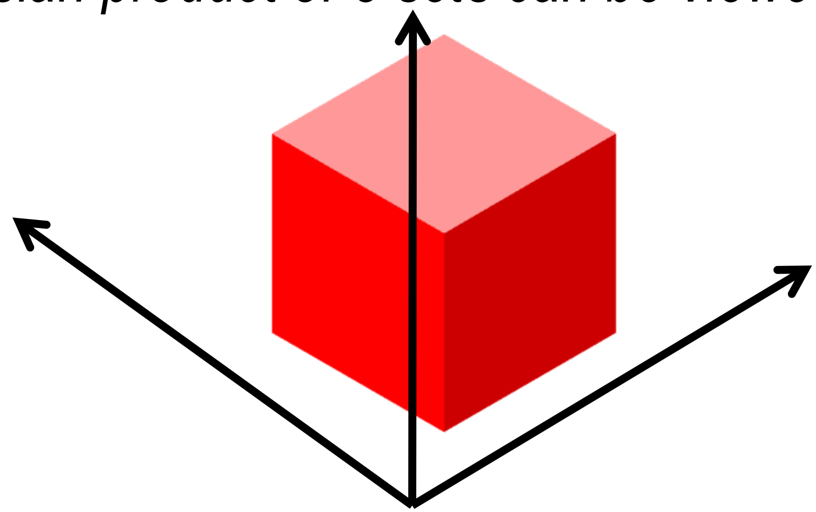
  $A \times B \times C = \{ (a,b,c) \mid a \in A,\ b \in B,\ c \in C\}$

  $A \times B \times C \times D = \{ (a,b,c,d) \mid a \in A,\ b \in B,\ c \in C,\ d \in D\}$

  $A_1 \times A_2 \times \ldots \times A_k = \{(a_1, a_2, \ldots, a_k) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_k \in A_k\}$

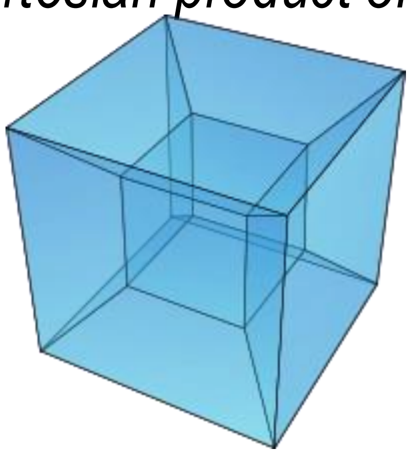
Cartesian Product of More Than Two Sets

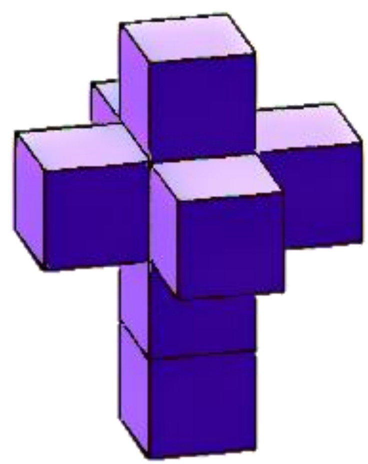
- *Cartesian product of 3 sets can be viewed as a `3-dimensional' set*



E.g. Coordinates (a,b,c) in R x R x R

- *Cartesian product of more than 3 sets is a multi-dimensional set*





More Relations (cntd)

- *Binary relations can be generalized to subsets of Cartesian products of more than two sets.*

- *A subset of the Cartesian product of 3 sets is a ternary relation*

  `x and y are parents of z' $\subseteq$ *People* $\times$ *People* $\times$ *People*

- *Any subset of the Cartesian product of k sets is a k-ary relation*

$$\{ (a_1, a_2, \ldots, a_k) \mid a_1 + a_2 + \ldots + a_k = 3 \} \subseteq \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$$

Relational Databases

- *A relational database is a collection of tables like*

No.	Name	Student ID	Supervisor	Thesis title
1.	Bradley Coleman	30101234	Petra Berenbrink	Algebraic graph theory
…	…	…	…	…

A table consists of a schema and an instance …

The instance of this table is a 5-ary relation, a subset of the Cartesian product

$$\mathbb{Z}^+ \times \text{Names} \times \text{8-strings\_of\_digits} \times \text{Names} \times \text{Meaningful\_Sentences}$$

Cartesian Product, Intersection and Union

- **Theorem**. *For any sets A,B,C*

  (1) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

  (2) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

  (3) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

  (4) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

- *Proof (of (2))*

$$\begin{aligned} A \times (B \cup C) &= \{ (a,b) \mid a \in A \wedge b \in B \cup C \} \\ &= \{ (a,b) \mid a \in A \wedge (b \in B \vee b \in C) \} \\ &= \{ (a,b) \mid (a \in A \wedge b \in B) \vee (a \in A \wedge b \in C) \} \\ &= \{ (a,b) \mid a \in A \wedge b \in B\} \cup \{ (a,b) \mid a \in A \wedge b \in C \} \\ &= (A \times B) \cup (A \times C) \end{aligned}$$

Q.E.D.

Sets, Relations, and Predicates

- *So all of predicate logic can be modelled in Set theory*

Unary predicate
$P(x)$

Set
$A = \{ x \mid P(x) \}$

Binary predicate
$P(x,y)$

Binary relation
$R = \{ (x,y) \mid P(x,y) \}$

Ternary predicate
$P(x,y,z)$

Ternary relation
$R = \{ (x,y,z) \mid P(x,y,z) \}$

Is this a good idea?

Everything everywhere all at once

- *Recall Russell's paradox U – the set of all sets which do not contain themselves*
- *Paradoxes are inconsistencies which lead to proofs of untrue facts*
- *Theorem: Let P be a logical formula. Then P is true*
- *Proof:*

Let $U = \{ S \mid S \notin S$ i.e. S is a set which does not contain itself $\}$

Case 1: $U \in U$

Then $U \notin U$ by definition, hence F is true

Case 2: $U \notin U$

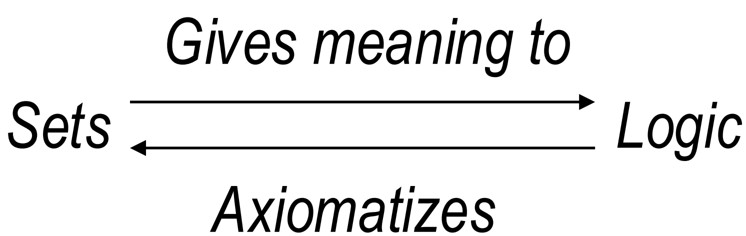
Then $U \notin U$ by definition, hence F is true

Since F is true, P by the law of contradiction.

Q.E.D.

A history lesson

- *Frege was writing his formalization of mathematics based on logic when Russell sent him the paradox in 1901*
- *The paradox worked because of assumed extensionality --- the existence of set-theoretic models which characterize predicates*
- *One method of fixing was by consistently axiomatizing set theory in logic (ZFC Set Theory), using universes (championed by Von Neumann) to avoid paradoxes*
- *The result is circular or impredicative:*



Intuitionism

- *Competing "fix" via Brouer was to reject logic altogether*
- *At the time, logic was a tool for discovering the objective truth of a mathematical statement*
- *Intuitionists were Nihilists who said that*
  "there is no objective truth, only a mathematician's intuition"
- *Meanwhile Russell disagreed with everyone:*
  "Logic exists, but should not be circular"
- *Heyting (1930's) committed blasphemy (at the time) and developed intuitionistic logic:*
  "The logic of what can be proved, not objective truth"

Intuitionistic logic

- *Formulas have no intrinsic truth*
- *P means "P is provable"*
- *$\neg$P is "unprovability" --- that is, provability of P implies a contradiction*
- *Law of the excluded middle is not valid:*
  P $\vee$ $\neg$ P is "P is provable, or the unprovability of P is provable"
- *Double negation is also not valid:*
  $\neg$ $\neg$ P means ???
- *BHK interpretation:*
  "Meaning" of a formula is A COMPUTER PROGRAM

Practice

Exercises from the Book:

7th edition: 27, 32, 38, 39 (page 126)

8th edition: 29, 34, 40, 41 (page 132 – 133)