Relations II

Discrete Mathematics

Courtesy of Andrei Bulatov

Previous Lecture

- *Cartesian product*
- *Properties of Cartesian products*
- *Relations*

Describing Binary Relations

- *Can also describe binary relations by a table or matrix*

  Matrix of a relation $R \subseteq A \times B$ is a rectangular table, rows of which are labeled with elements of A (in any but fixed order), and columns are labeled with elements of B. We write 1 in the intersection of row a and column b if and only if $(a,b) \in R$; otherwise we write 0.

Brotherhood

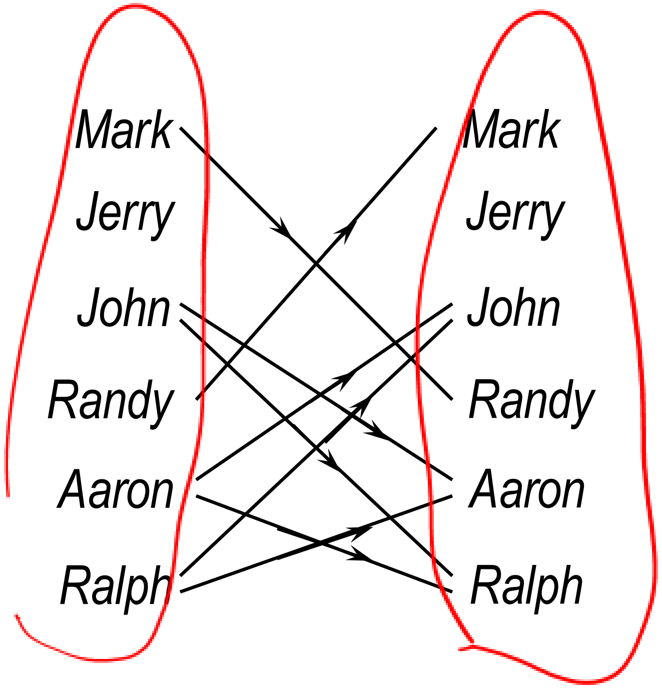
$R \subseteq P \times P$

$$\begin{array}{c|cccccc}
 & \text{Mark} & \text{Jerry} & \text{John} & \text{Randy} & \text{Aaron} & \text{Ralph} \\
\hline
\text{Mark} & 0 & 0 & 0 & 1 & 0 & 0 \\
\text{Jerry} & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{John} & 0 & 0 & 0 & 0 & 1 & 1 \\
\text{Randy} & 1 & 0 & 0 & 0 & 0 & 0 \\
\text{Aaron} & 0 & 0 & 1 & 0 & 0 & 1 \\
\text{Ralph} & 0 & 0 & 1 & 0 & 1 & 0
\end{array}$$

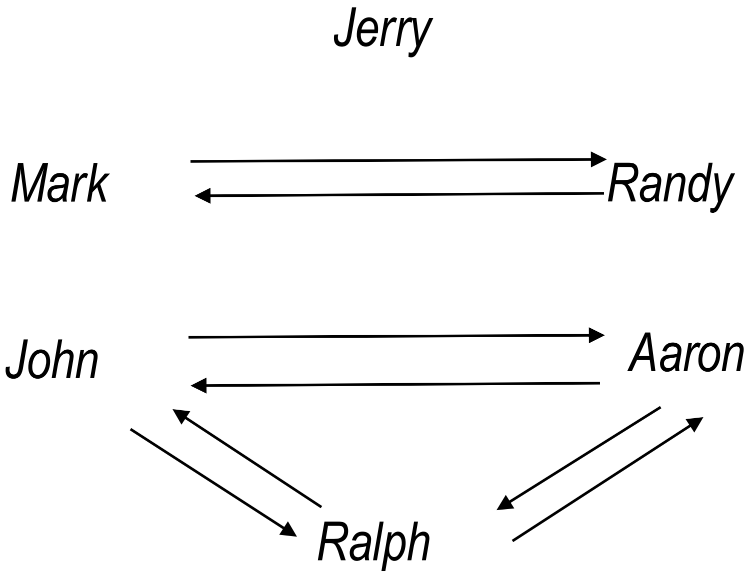
Describing Binary Relations (cntd)

- *Binary relations can also be described by a graph*

Graph of a relation $R \subseteq A \times B$ *consists of two sets of vertices labeled by elements of* A *and* B*. A vertex* a *is connected to a vertex* b *with an edge (arc) if and only if* $(a,b) \in R$*.*



If $A = B$ *then we may use only one set of vertices*

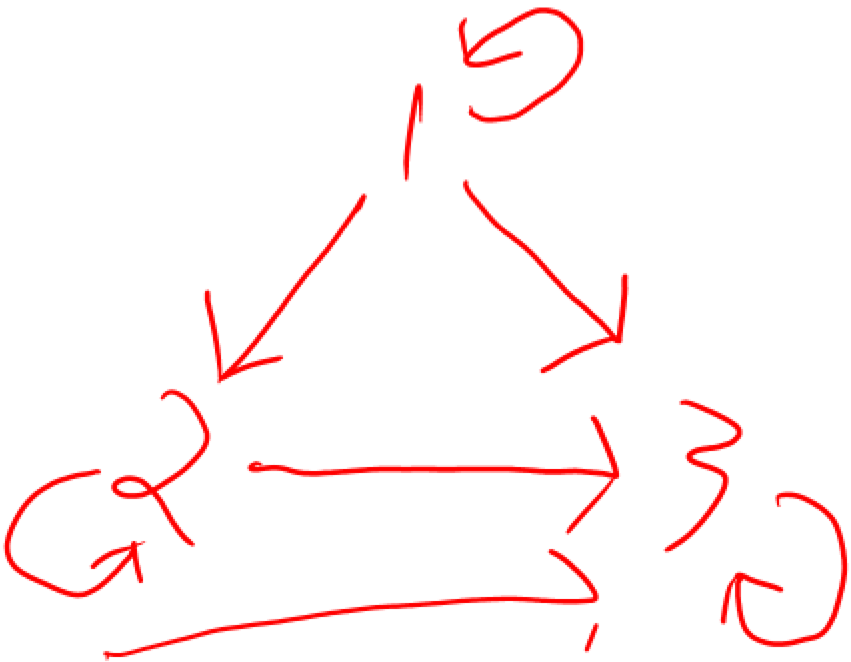


Example

- *Describe the relation* $R(a,b) = \{ (a,b) \mid a,b \in \{1,2,3\} \text{ and } a \le b \}$

$R(a,b) = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

	1	2	3
1	1	1	1
2	0	1	1
3	0	0	1



Properties of Binary Relations – Reflexivity

- *A binary relation* $R \subseteq A \times A$ *is said to be* *reflexive* *if* $(a,a) \in R$ *for all* $a \in A$.

Example:

$(a,b) \in R \subseteq \mathbb{Z} \times \mathbb{Z}$ *if and only if* $a \le b$

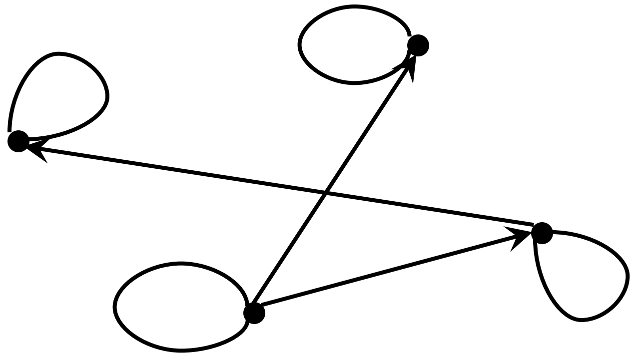
This relation is reflexive, because $a \le a$ *for all* $a \in \mathbb{Z}$

Matrix:

$$\begin{pmatrix} 1 & * & * & * \\ * & 1 & * & * \\ * & * & 1 & * \\ * & * & * & 1 \end{pmatrix}$$

1's on the diagonal

Graph:



Loops at every vertex

Properties of Binary Relations – Transitivity

- *A binary relation* $R \subseteq A \times A$ *is said to be* ***transitive*** *if, for any* $a,b,c \in A$, *if* $(a,b) \in R$ *and* $(b,c) \in R$ *then* $(a,c) \in R$.

Example:

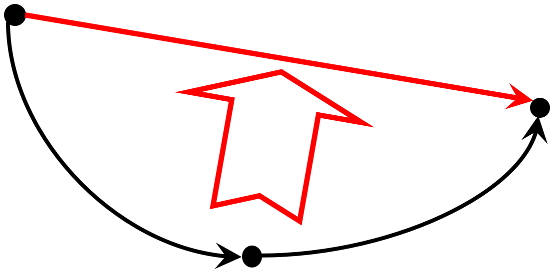
$(a,b) \in R \subseteq \mathbb{Z} \times \mathbb{Z}$ *if and only if* $a \leq b$

This relation is again transitive, because if $a \leq b$ *and* $b \leq c$, *then* $a \leq c$ *for all* $a,b,c \in \mathbb{Z}$

Matrix:

?

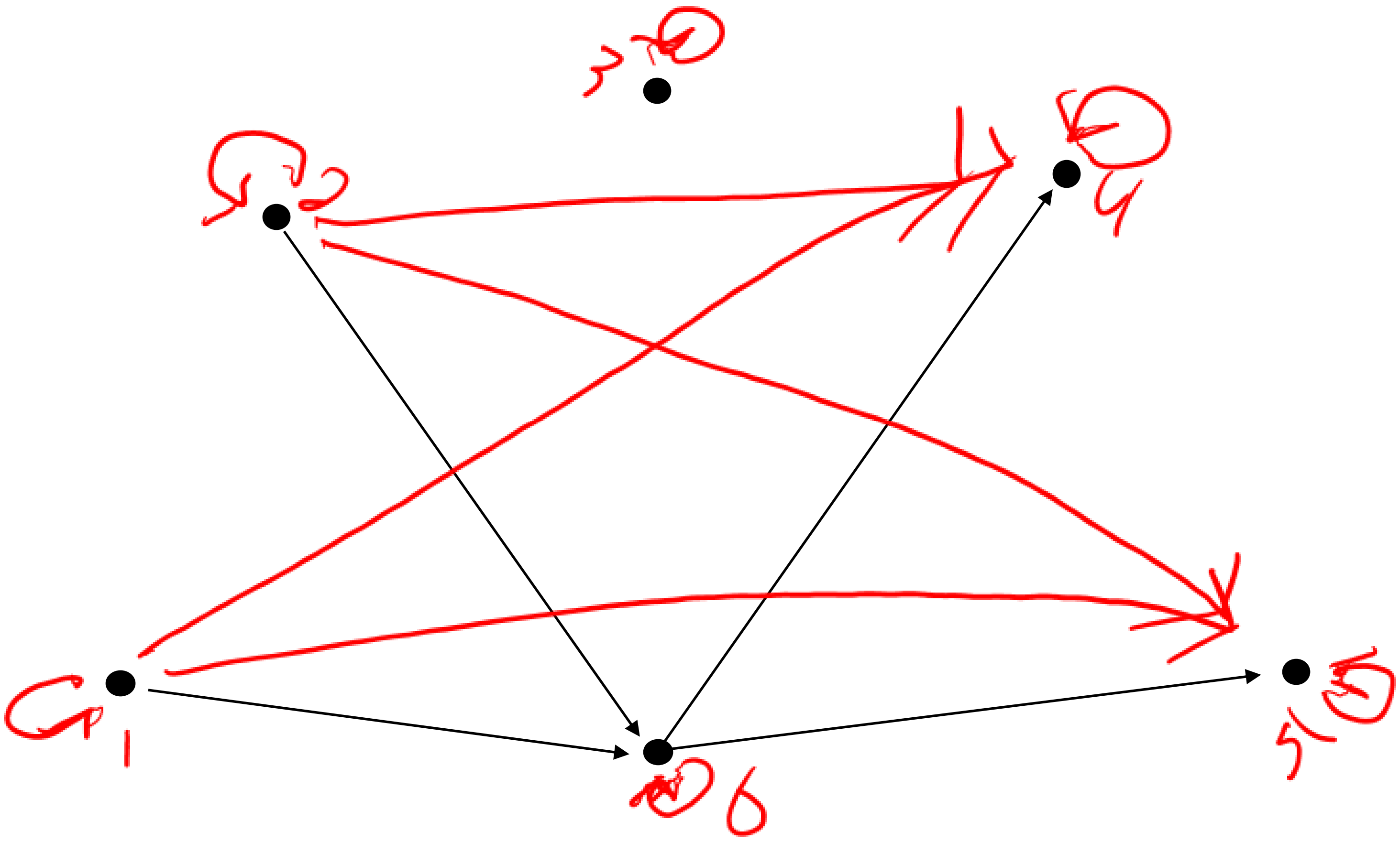
Graph:



Edges compose

Exercise

- *Make the following graph reflexive and transitive*



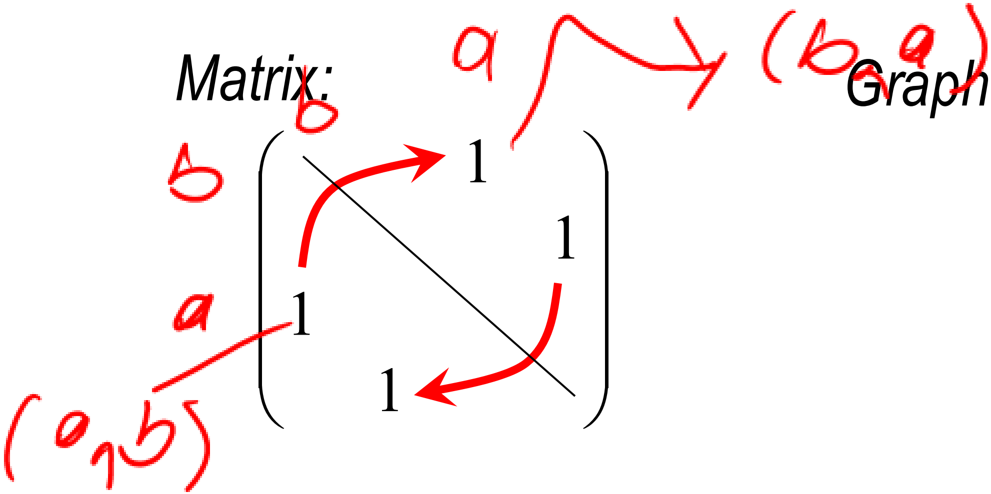
Properties of Binary Relations – Symmetricity

- *A binary relation $R \subseteq A \times A$ is said to be* ***symmetric*** *if, for any $a,b \in A$, if $(a,b) \in R$ then $(b,a) \in R$.*

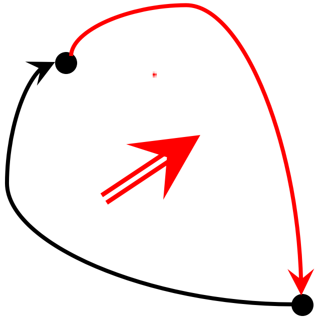
Example:

The relation a = b is symmetric because if a = b, clearly also b = a

Matrix:



Graph:



Matrix is symmetric w.r.t. the diagonal

Every edge has an opposite

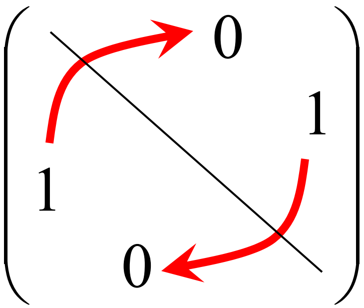
Properties of Binary Relations – Anti-Symmetricity

- *A binary relation* $R \subseteq A \times A$ *is said to be* *anti-symmetric* *if, for any* $a,b \in A$, *if* $(a,b) \in R$ *and* $(b,a) \in R$ *then* $a = b$.

Example:

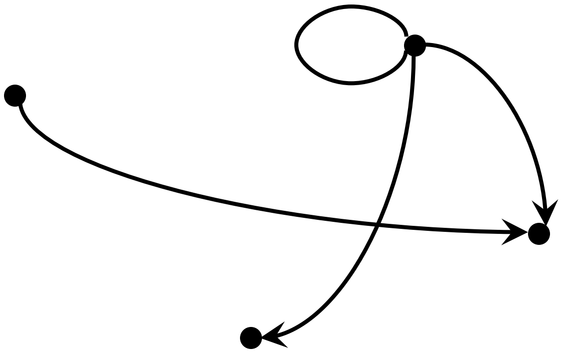
The relation a < b is anti-symmetric because if a < b, b cannot also be less than a

Matrix:



Matrix is anti-symmetric w.r.t. the diagonal

Graph:



No edges have an opposite

Examples

	reflexive	symmetric	transitive	anti-symmetric
Brotherhood x is a brother of y	It depends	Yes (on men) No (on people)	No (with half brothers) Yes (otherwise)	No
Neighborhood x is a neighbor of y	It depends	Yes	No	No
$x \leq y$	Yes	No	Yes	Yes
$x = y$	Yes	Yes	Yes	No

Equivalence relations

- *A binary relation R on a set A is said to be an* **equivalence relation** *if it is reflexive, symmetric, and transitive.*
- *Example:*

  Let $R = \{(a,b) \subseteq \text{People} \times \text{People} \mid a \,\&\, b \text{ are the same age.}\}$

  Reflexive? ✓
  Symmetric? ✓
  Transitive? ✓

- *Example:*

  Let $R = \{(a,b) \mid a \leq b.\}$

  Reflexive? ✓
  Symmetric? ✗
  Transitive? ✓

Equivalence Classes

- *Take $a \in A$ and R an equivalence relation on A.*
- *The set $C(a) = \{ b \mid (a,b) \in R\}$ is the* ***equivalence class*** *of a.*
- *Informally, $C(a)$ is "all elements which are equal to a wrt R"*

- *Example:*

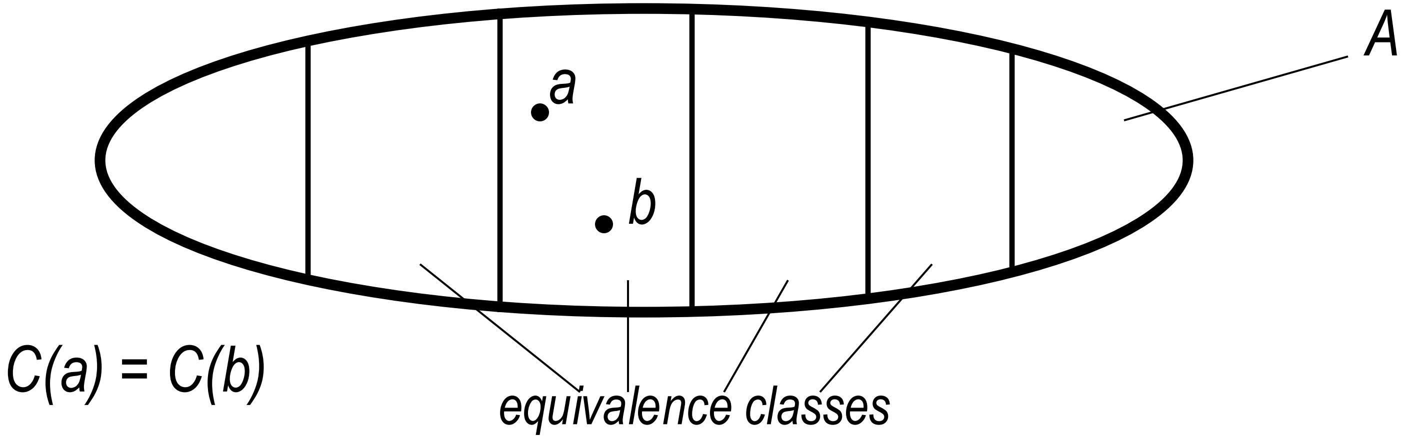
  Let $R = \{(a,b) \subseteq \text{People} \times \text{People} \mid a \;\&\; b \text{ are the same age.} \}$

  $C(\text{Isaac}) = \{ b \mid b \text{ is 3 years old} \}$

  $C(\text{Elliot}) = \{ b \mid b \text{ is 1 year old} \}$

Partitions

- *Thus the equivalence classes divide up the set A into disjoint subsets.*



$C(a) = C(b)$

- *A collection of subsets $\underline{M_1, \ldots, M_n}$ of a set A is called a* **partition** *if the following conditions hold.*

  (1) Every $M_i \neq \varnothing$

  (2) If $M_i \neq M_j$ then $M_i \cap M_j = \varnothing$

  (3) $A = \bigcup_{i=1}^{n} M_i$

Partitions and Equivalence Relations

- **Theorem**. *Let A be a set.*

  (1) If R is an equivalence relation on A, then its equivalence classes form a partition of A.

  (2) If $M_1,\ldots,M_n$ is a partition of the set A, then the relation R defined as follows: $\underline{(a,b) \in R}$ if and only if $a,b \in M_i$ for some M_i, is an equivalence relation on A.

- *Proof*

  (1) Next slide

  (2) Homework

Equivalence classes partition a set

- **Let R be an equivalence relation on A. We need to show**
  - (1) *For any* $a \in A$, *the class* $C(a) \neq \varnothing$
  - (2) *If* $C(a) \neq C(b)$ *then* $C(a) \cap C(b) = \varnothing$
  - (3) $A = \bigcup_{a \in A} C(a)$
- **Proof**
  - (1) R *is reflexive, therefore,* $(a,a) \in R$. *Hence* $a \in C(a) \neq \varnothing$
  - (3) *Clearly* $\underline{\bigcup_{a\in A} C(a)} \subseteq \underline{A}$ *because each* $C(a) \subseteq A$

    Now suppose $a \in A$. *Then* $a \in C(a)$, *so* $a \in \bigcup_{a\in A} C(a)$

    and hence also $A \subseteq \bigcup_{a\in A} C(a)$

Equivalence classes partition a set (cntd)

Proof of (2): *If $C(a) \neq C(b)$ then $C(a) \cap C(b) = \varnothing$*

Suppose $c \in C(a) \cap C(b)$ to prove by contrapositive

We need to show that $C(a) = C(b)$.

First we show that $(a,b) \in R$:

Since $c \in C(a) \cap C(b)$, we have $(a,c), (b,c) \in R$.

Since R is symmetric, $(c,b) \in R$.

Since R is transitive, $(a,b) \in R$.

Now we show $C(a) \subseteq C(b)$

Let $x \in C(a)$. Then $(x,a) \in R$

But $(a,b) \in R$, so by transitivity, $(x,b) \in R$

Hence $x \in C(b)$

Remaining case $C(b) \subseteq C(a)$ is similar

Q.E.D.

Example: congruences (more in a few weeks)

- *Let k be an integer. Integers a,b are* congruent modulo k, *denoted* $a \equiv b \pmod{k}$, *if their remainders when they are divided by k are equal, or, equivalently, if k divides a – b.*

… -3, 0, 3, 6, … are congruent modulo 3,
and so are …, -4, -1, 2, 5, … and …, -5, -2, 1, 4, …

- *The relation* $\equiv \pmod{k}$, *`to be congruent modulo k' is*
  - *reflexive, because k divides* $a - a = 0$
  - *symmetric, because if k divides* $a - b$ *then it also divides* $b - a$
  - *transitive, because if k divides* $a - b$ *and* $b - c$, *then it also divides* $a - c = (a - b) + (b - c)$
- $\equiv \pmod{k}$, *is an equivalence relation with equivalence classes* $\{ a \mid \text{there is } b \text{ with } a = bk + c\}$
- *Arithmetic on these classes is called* modular arithmetic