# Orders

*Discrete Mathematics*

*Courtesy of Andrei Bulatov*

# Previous Lecture

- *Describing binary relations*
- *Types of relations*
- *Equivalence relations*

# Types of binary relations

- **Reflexive**

  *$(a,a) \in R$ for all $a \in A$.*

- **Symmetric**

  *If $(a,b) \in R$ then $(b,a) \in R$.*

- **Transitive**

  *If $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.*

- **Anti-symmetric**

  *If $(a,b) \in R$ and $(b,a) \in R$ then $a = b$.*

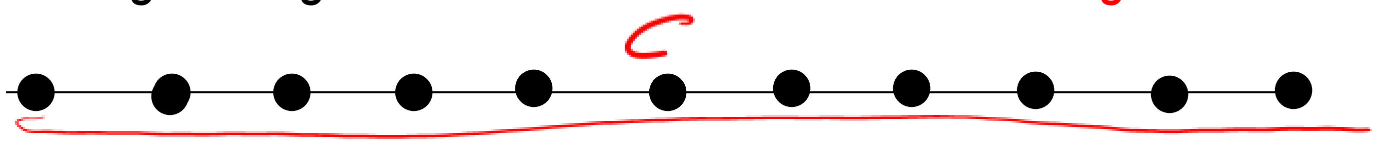
- **Equivalence**

  *If R is reflexive, symmetric, and transitive*

# Equivalence relations

- *Recall that equivalence relations generalize the notion of equality*
  - *For every a, a is equal to itself (reflexive)*
  - *For every a and b, a and b are equal if b and a are equal (symmetric)*
  - *For every a,b, and c, if a and b are equal and b and c are equal, a and c should be equal (transitive)*

- *How might we generalize the notion of an ordering?*



*If a comes before b, what can (or can't) be true?* $b \not< a$

*If a comes before b and b comes before c, what do we know?* $a < c$

*What about a? Should it be "before" itself? "before or equal?"*

# Orders

- *A relation $R$ on a set $A$ is called a partial order if it is* **reflexive**, **transitive**, **anti-symmetric**.
- *If R is instead* **antireflexive**, *then it is a strict partial order*

*Examples:*

- $a \leq b$ *on the set of real numbers is a partial order*
- $a < b$ *on the set of real numbers is a strict partial order*
- *is $a > b$ a (strict) partial order?*

- *A set S together with a partial order R is called a poset (partially ordered set)*

# Subsets

- **Claim**: *R* = {(*a*,*b*) ∈ $\mathcal{P}$(A) x $\mathcal{P}$(A) | *a* ⊆ *b* } *is a partial order*

*Reflexive?* $\forall a \in P(A),\ a \subseteq a$ ✓

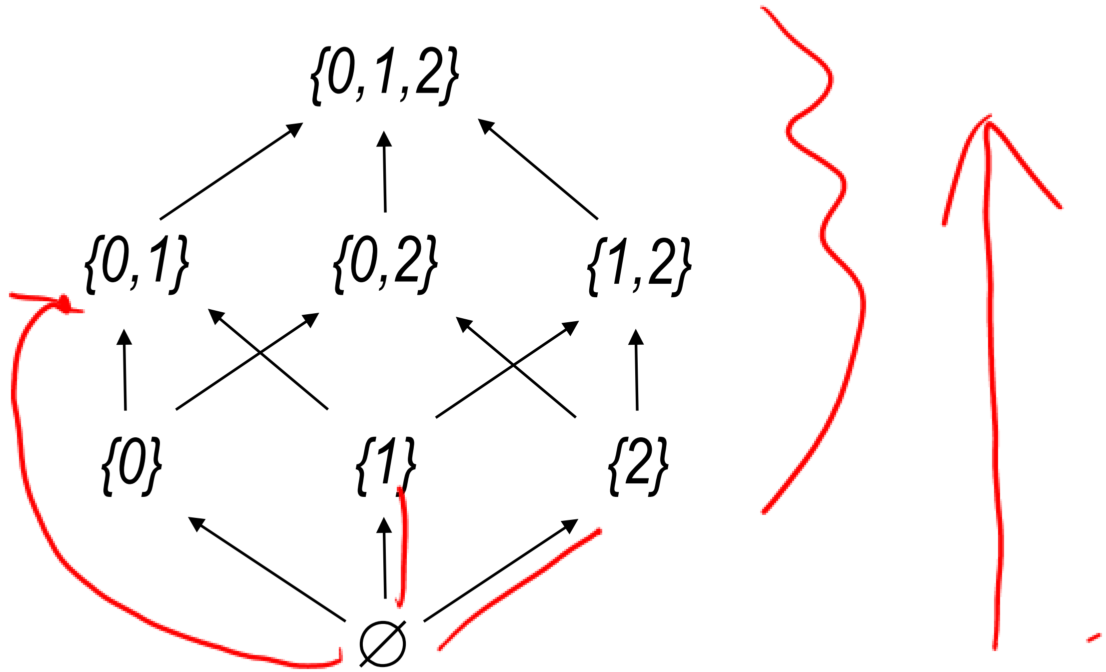
*Transitive?* $a \subset b,\ b \subseteq c \Rightarrow a \subseteq c$ ✓

*Anti-symmetric?*

$a \subseteq b \wedge b \subseteq a \Rightarrow a = b$ ✓

# Hasse diagrams

- *Standard way of drawing or visualizing a partial order is via a Hasse diagram*




- *Idea is to only show enough orderings to reconstruct the whole order*
  - *Do not need to show reflexive edges*
  - *Do not need to show any edges implied by transitivity*
  - *Put elements higher in the order higher on the page, by anti-symmetry*

# Constructing Hasse diagrams

- *Start with the whole order*
- *Remove any reflexive edges*
- *Remove any edges implied by transitivity*

$$
\begin{tikzcd}
 & \{0,1\} & \\
\{0\} \arrow[ur] & & \{1\} \arrow[ul] \\
 & \varnothing \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

# Divisibility

- **Definition**: *Given integers a and b, we say that a divides b (denoted a | b) if and only if b = a*k for some integer k*

  *Does 3 | 6?* ✓ $6 = 3 \cdot 2$ *Does 3 | 7?* ✗

- **Claim**: *R = {(a,b) | a divides b } is a partial order*

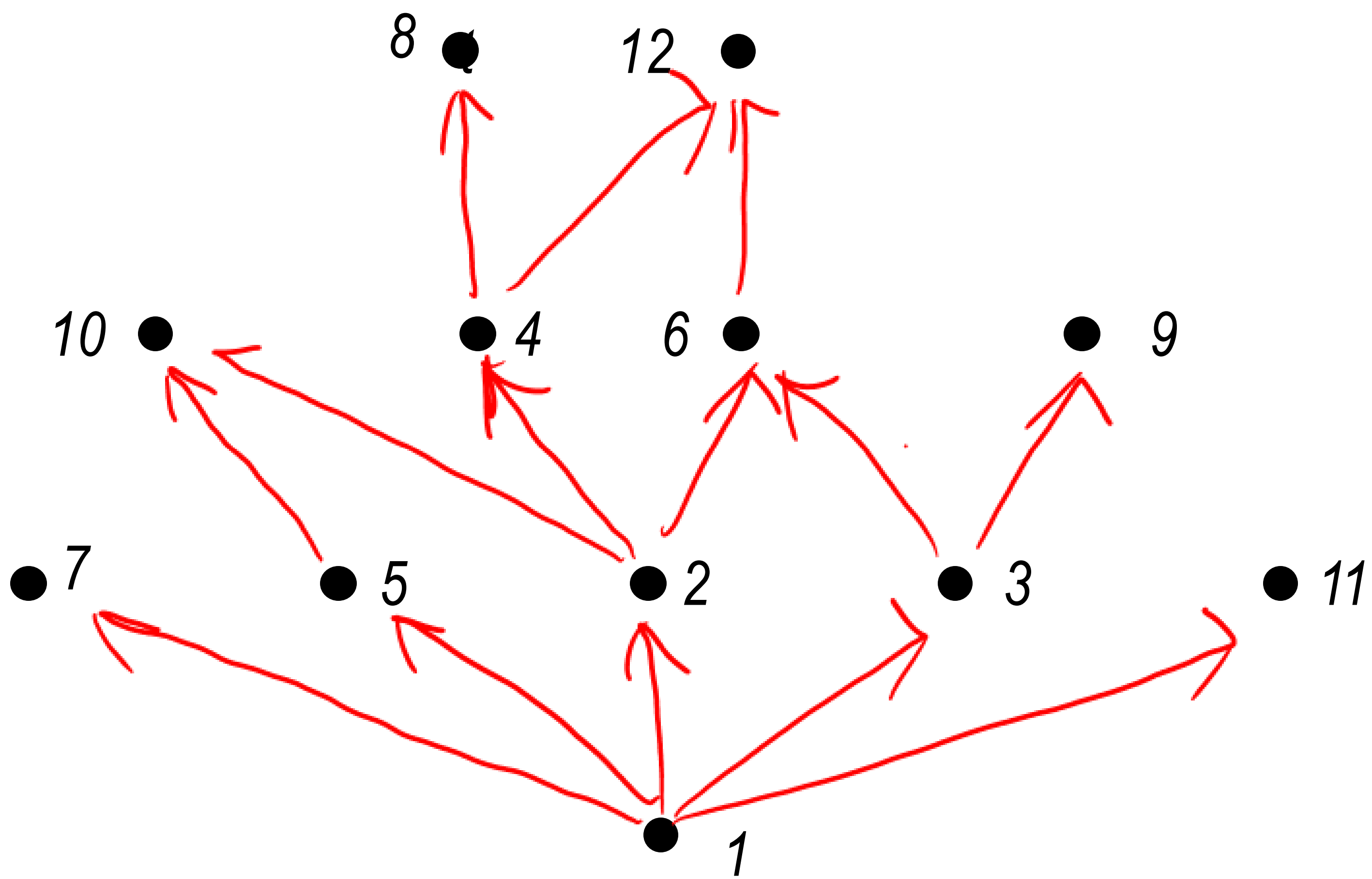
  *Reflexive?* $a|a \Leftrightarrow a = a * k$ ✓

  *Transitive?* $a|b \Rightarrow b = a \cdot k$
  $b|c \Rightarrow c = b \cdot l = a \cdot k \cdot l$ ✓

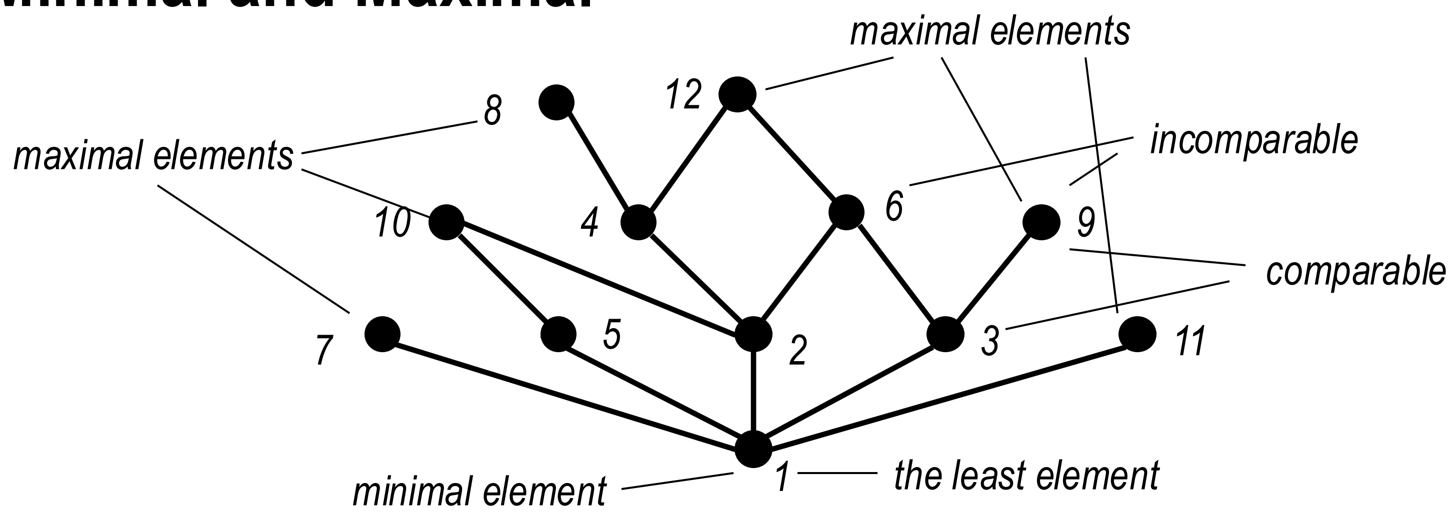
  *Anti-symmetric?*
  $a|b \Rightarrow b = a \cdot k = b = b \cdot l \cdot k$
  $b|a \Rightarrow a = b \cdot l$ $l = 1$ $k = 1$

# Divisibility, visualized

- *Draw the divisibility order on {1,2,…,12}*



# Minimal and Maximal

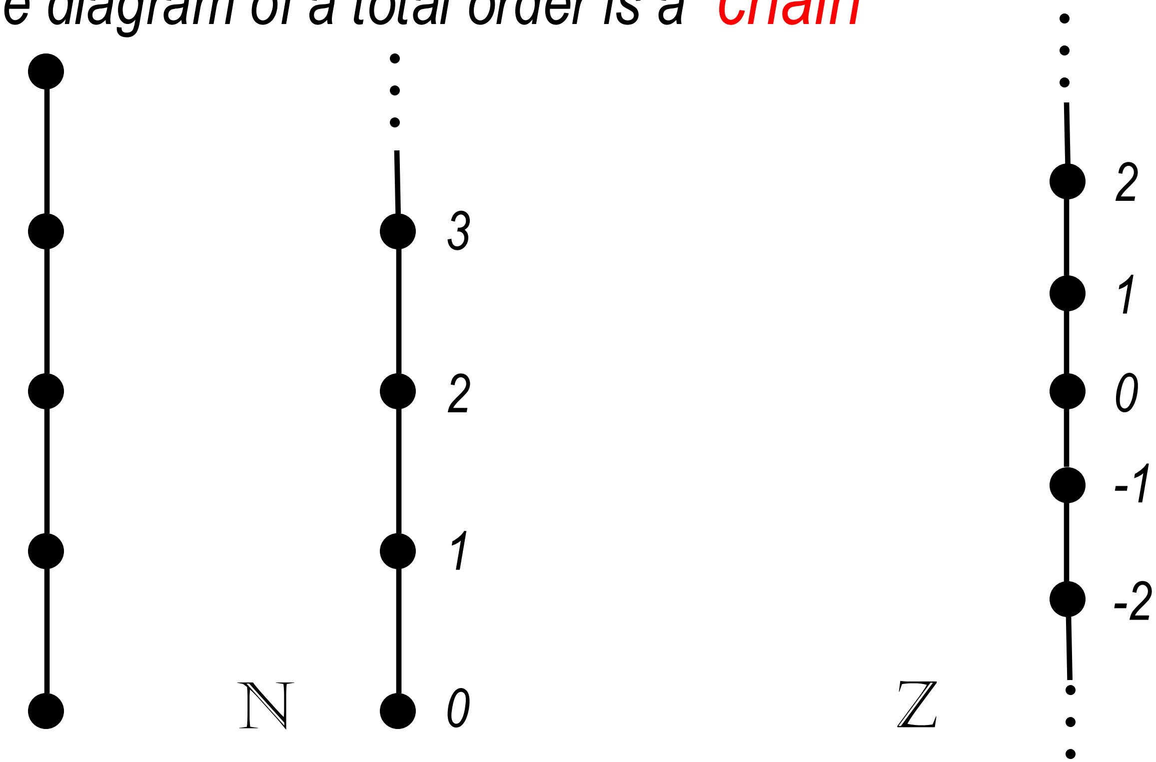



- *Elements $a,b$ are said to be comparable if $(a,b) \in R$ or $(b,a) \in R$*
- *Otherwise they are called incomparable*
- *Element $a$ is minimal if for any $b$ if $(b,a) \in R$ then $a = b$*
- *Element $a$ is maximal if for any $b$ if $(a,b) \in R$ then $a = b$*
- *Element $a$ is called the least element if for any $b$, $(a,b) \in R$*
- *Element $a$ is called the greatest element if for any $b$, $(b,a) \in R$*

# Total Order

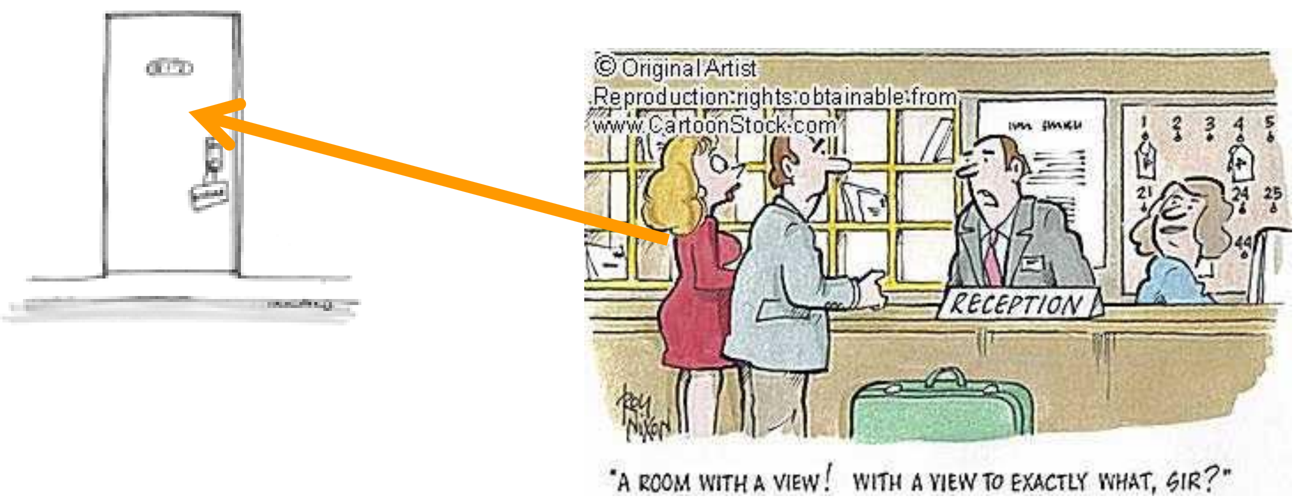
integers $\geq 0$

- *A partial order is* total *if every two elements are comparable*
- *Sets* $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ *are totally ordered with respect to* $\leq$
- *The power set of a set is* not *a total order with respect to* $\subseteq$
- *The diagram of a total order is a* chain



# Functions

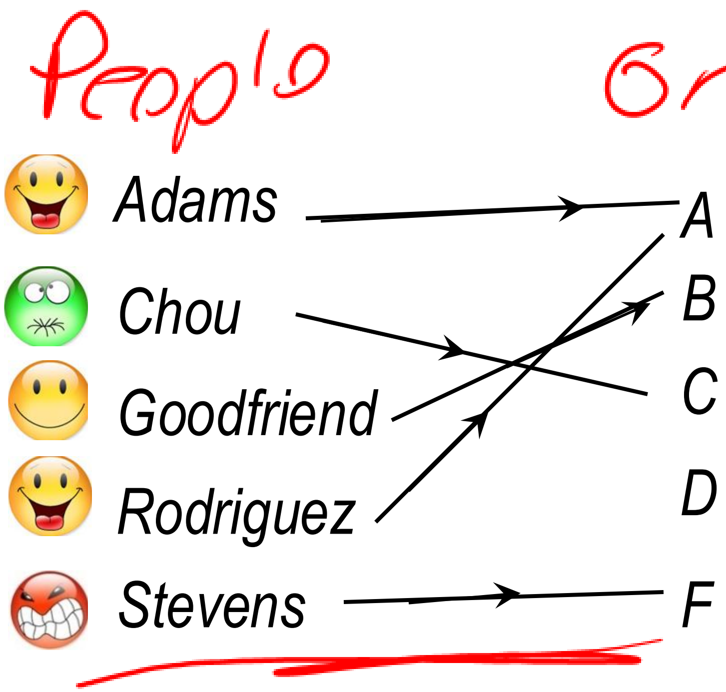
- *Another important type of relation is a function*
- *Functions assign elements of one set to elements of another*
- *For example, assign rooms to people in a hotel*




- *Or we may assign a grade to each student from a class*
- *What we get is a set of pairs (Person, Door) or (Student, Grade), i.e. a relation, but a very particular one*

# Functions (cntd)

- *A relation $R$ from $A$ to $B$ is called a* **function** *from $A$ to $B$, if for every $a \in A$ there is* **exactly one** *$b \in B$ such that $(a,b) \in R$.*
  - *Also called mappings or transformations*



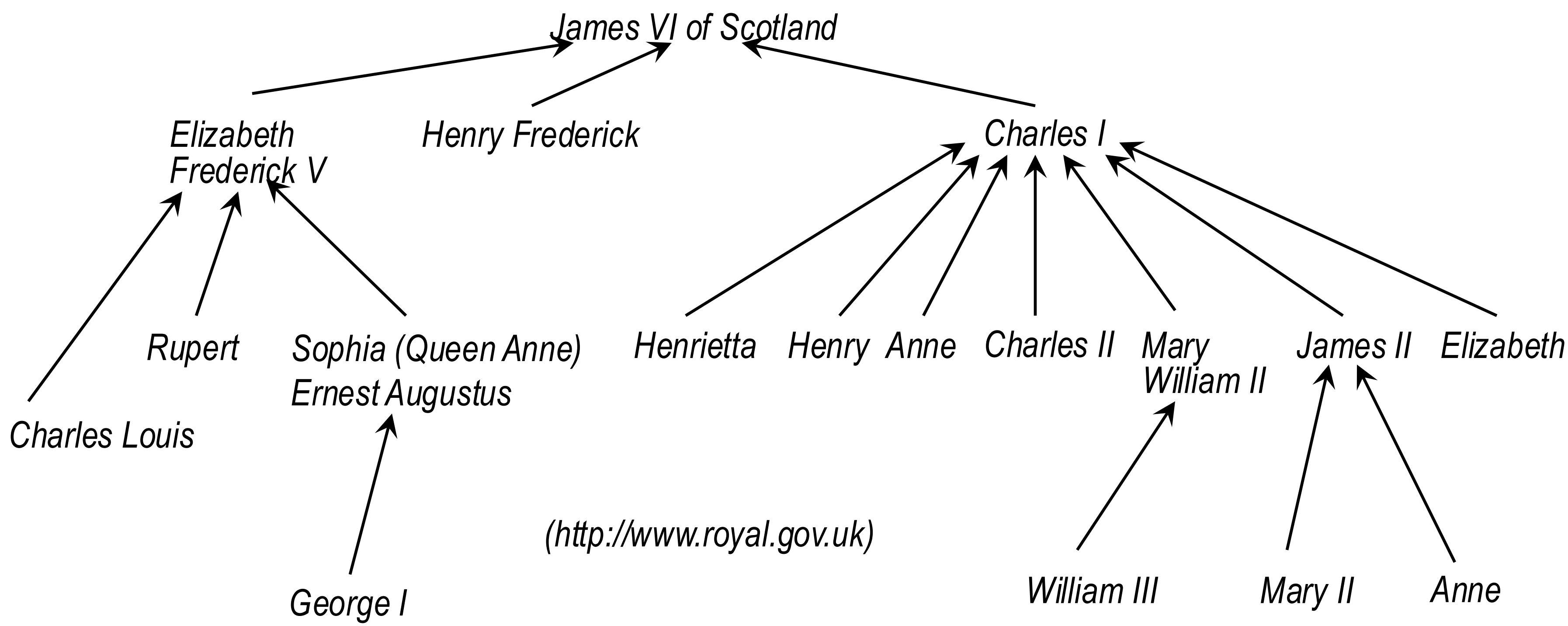

*We use $f,g,h$ to denote functions*

$f: A \to B$ $\qquad f(a) = b$

$f(\text{Rodriguez}) = A$

# Example

- *Consider the relation R = {(a,b) | b is the (biological) father of a}*




- *Is R a function?*
- *What about the relation {(a,b) | a and b are brothers}?*

# Describing Functions

- *A function is a relation, therefore we can use all methods of describing relations.*
- *Functions don't have many tuples though, so often use a table to list each pair of the relation instead.*

| Student | Grade |
|---|---|
| Adams | A |
| Chou | C |
| Goodfriend | B |
| Rodriguez | A |
| Stevens | F |

v.s.

| | A | B | C | D | F |
|---|---|---|---|---|---|
| Adams | 1 | | | | |
| Chou | | | 1 | | |
| Goodfriend | | 1 | | | |
| Rodriguez | 1 | | | | |
| Stevens | | | | | 1 |

One 1 per row in matrix

# Describing Functions (cntd)

- *Numerical functions can be computed using a formula*

$$f(x) = x^2$$

- *The most general way is to use some algorithm to compute a function*

*`The letter grade is A, if the numerical mark is in between 100 and 85; the letter grade is B, if …*

*Functions in programming languages:*

*int* **floor***(float real) {…}* *in Java*

**function** *floor(x:* **real***):* **integer** *in Pascal*

# Practice

- *Are the following relations reflexive? symmetric? transitive? anti-symmetric?*
    - *Motherhood: `x is the mother of y'*
    - *Intersect: `straight lines x and y intersect'*

- *Show that equivalence of propositional formulas is an equivalence relation on the set of propositional formulas*

- *Define a relation R on propositions such that (p,q) in R if and only if p → q. Is this relation an equivalence? Is it an order? What if R is defined on* **equivalence classes** *of propositional formulas --- i.e. R is a relation on the set S = { C(p) | p is a propositional formula } where C(p) is the set of propositional formulas equivalent to p, and (C(p),C(q)) in R if p → q?*