Functions

Discrete Mathematics

Courtesy of Andrei Bulatov

Previous Lecture

- *Partial orders*
- *Diagrams of partial orders*
- *Functions (intro)*
  - *$f: A \to B$ is a relation R such that $(a,b) \in R$ and $(a,c) \in R$ implies $b = c$*

Domain and Codomain

- *Let f: A → B be a function from A to B. Then A is called the* **domain** *of f, and B is called the* **codomain** *of f.*

domain

codomain

- *If f(a) = b, then b is called an* **image** *of a, and a is called a* **preimage** *or* **fiber** *of b.*
- *How many images can a have via f?* 1
- *How many preimages can b have?* any number

Range and images

- *Given a subset $C \subseteq A$ of the domain A, the image of C is the subset of the codomain consisting of images of elements of C:*

  $$f(C) = \{ b \in B \mid \exists c \in C \;\; f(c) = b \}$$

- *The range of f is the image of A itself:*

  $$range(f) = f(A) = \{ b \in B \mid \exists a \in A \;\; f(a) = b \}$$

	A	B	C	D	F
Adams	1				
Chou			1		
Goodfriend		1			
Rodriguez	1				
Stevens					1

domain = { Adams, Chou, Goodfriend, Rodriguez, Stevens }

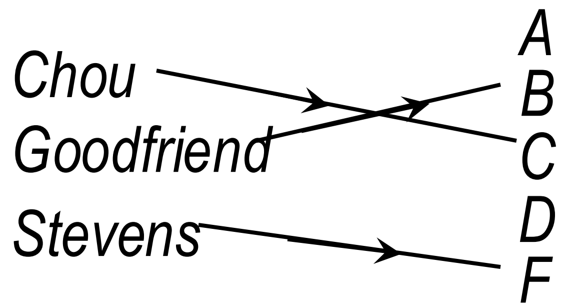
codomain = { A, B, C, D, F }

range = { A, B, C, F }

f({Adams,Chou}) = { A, C }

Restrictions of functions

- *Let $f: A \to B$ be a function and $C \subseteq A$. A function $f|_C : C \to B$ is called a* **restriction** *of f to C if $f|_C(a) = f(a)$ for all $a \in C$*
- *Example: Let C = {Chou, Goodfriend, Stevens}. Then*



is a restriction of f

- *Example:*

  Let $f:\mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2$

  Then $f|_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{R}$ is the function $f(x) = x^2$ on integers

Extensions of functions

- *Let* $C \subseteq A$ *and* $f: C \to B$. *Any function* $g: A \to B$ *such that* $g(a) = f(a)$ *for all* $a \in C$ *is called an* **extension** *of* f.

$g|_C = f$

- *Example: Consider* $f: \mathbb{Z} \to \mathbb{Z}$ *is defined as follows:*

$$f(a) = a$$

Let g *be the* **floor function***:*

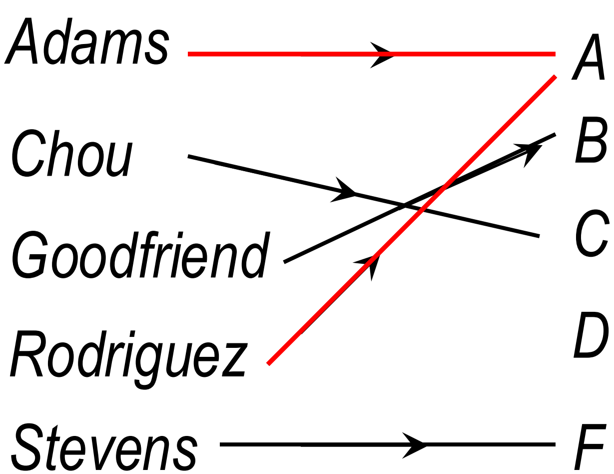
$$g(x) = \lfloor x \rfloor = \text{the greatest integer less than or equal to } x$$

Clearly, $g: \mathbb{R} \to \mathbb{Z}$, *and* $g(a) = a = f(a)$ *for any integer* a.

Thus, g *is an extension of* f

One-to-One Functions

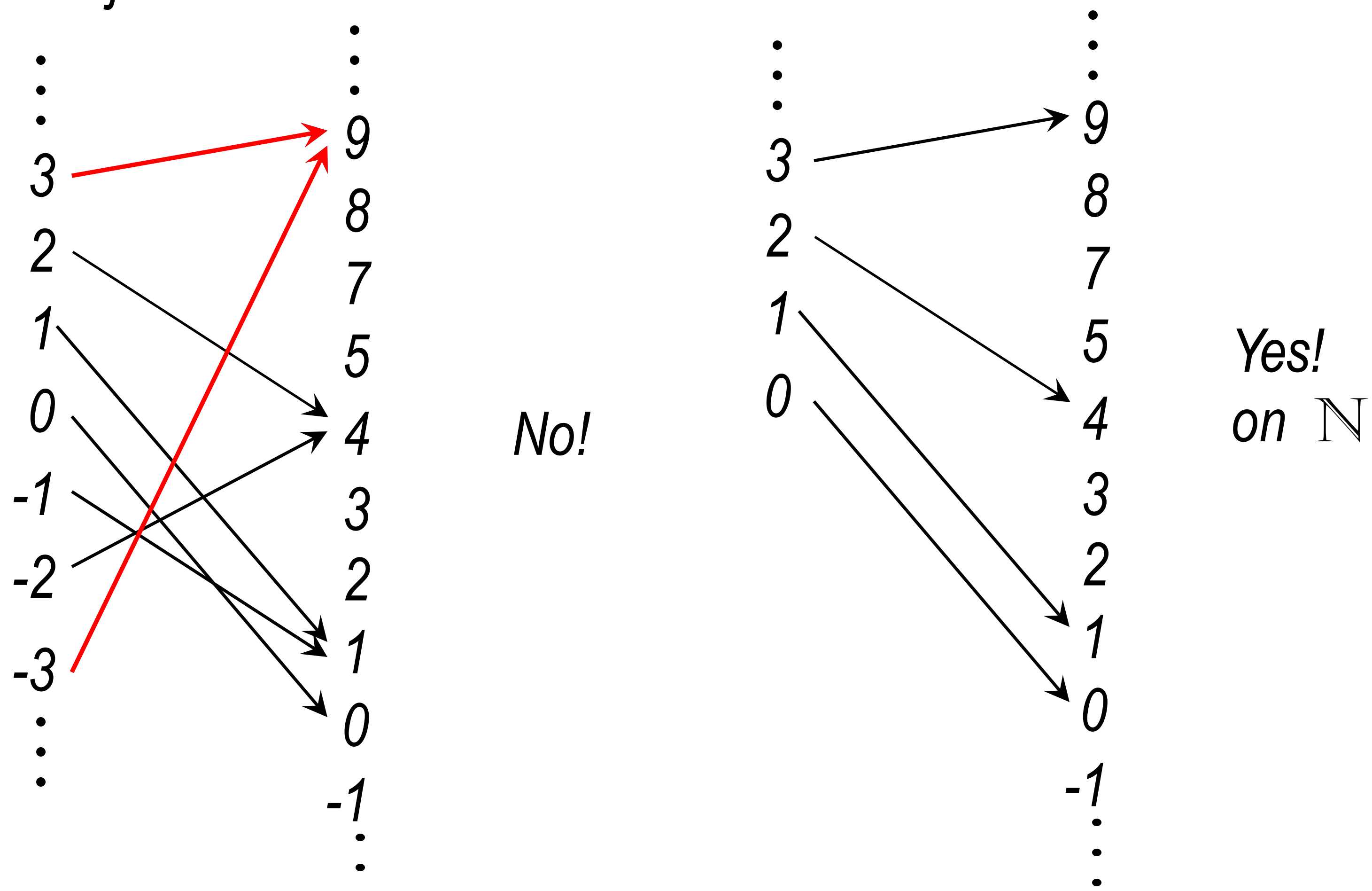
- A function f is said to be one-to-one, or injective, if and only if $f(a) = f(b)$ implies $a = b$ for all a,b in the domain.
- Contrapositive is if $a \neq b$ then $f(a) \neq f(b)$ – i.e., no two distinct elements have the same image.
- Is this function injective?



No!

One-to-One Functions (cntd)

- *Let's consider the function $f(x) = x^2$ on $\mathbb{Z}$*
  Is it injective?



Onto Functions

- *A function f from A to B is called* **onto**, *or* **surjective**, *if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.*
- *Symbolically:* $\forall b \, \exists a \, (f(a) = b)$
- *A function is called a* **surjection** *if it is onto.*
- *Examples:*
  - $f(x) = x + 1$?

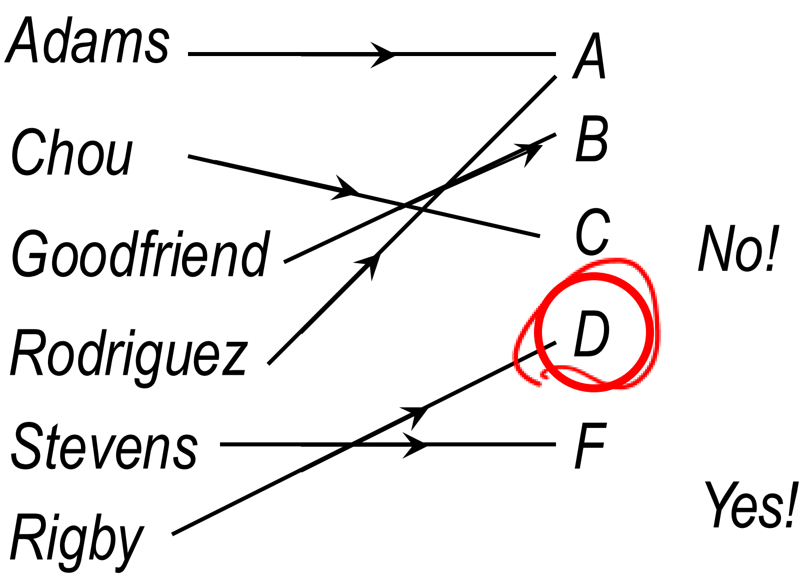
    Let $b \in \mathbb{Z}$. Then $f(b-1) = b$
  - $f(x) = x^2$ on $\mathbb{Z}$?

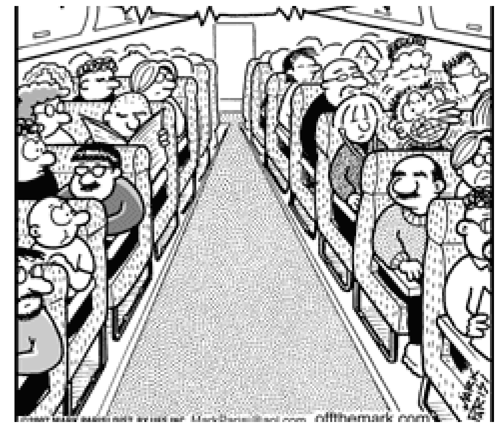
    consider $2 \in \mathbb{Z}$. $\nexists x$ s.t. $x^2 = 2$
  - $f(x) = x^2$ on $\mathbb{R}^+$?

    Let $b \in \mathbb{R}^+$. Then $f(\sqrt{b}) = b$

Onto Functions (cntd)

- *More examples*



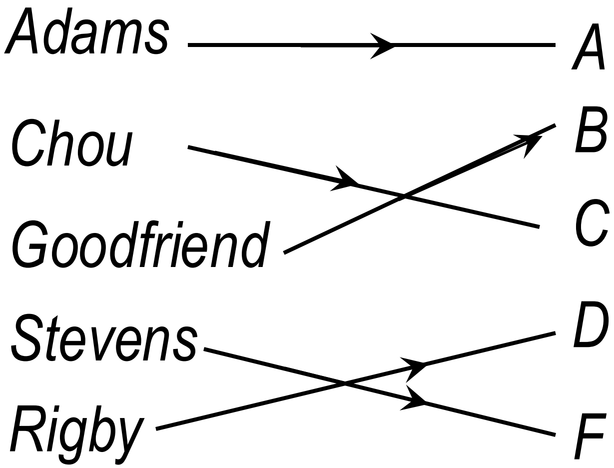


This mapping was onto!

- *f(a) = b if b is the father of a*

Bijections

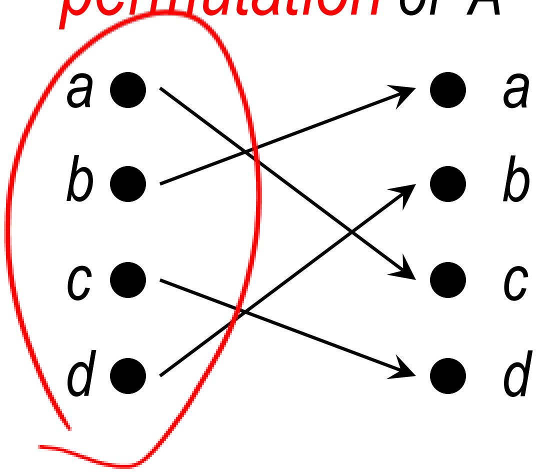
- *A function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto.*

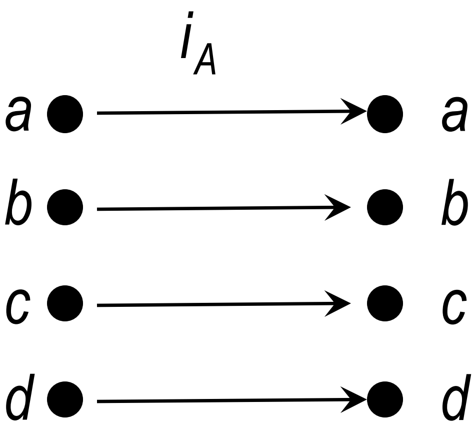


- *If there is a bijection from a set A to a set B, then these sets in a certain sense are equal or identical.*

Bijections (cntd)

- *Numerical functions:*
  - $f(x) = x + 1$ is a bijection on $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, but not on $\mathbb{N}$
  - $f(x) = x^2$ is a bijection on $\mathbb{R}^+$, but is not on any other numerical set
- *A bijection from a set A to the same set A is called a* *permutation* *of A*



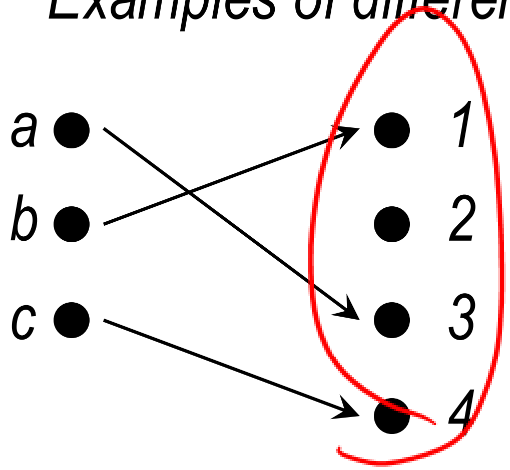


- *The* *identity function* *on a set A is the function* $i_A : A \to A$, *where*

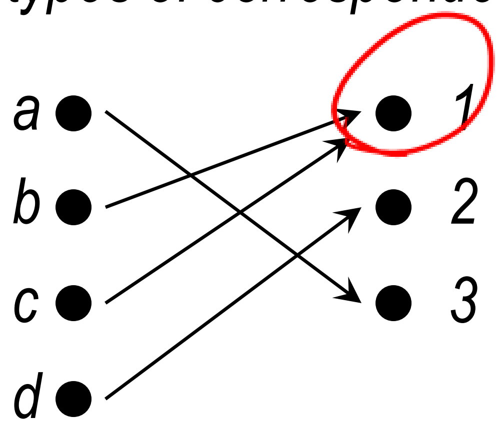
$$i_A(x) = x$$

Functions and Properties

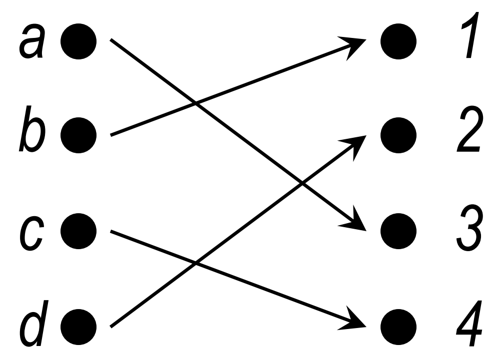
- *Examples of different types of correspondences*



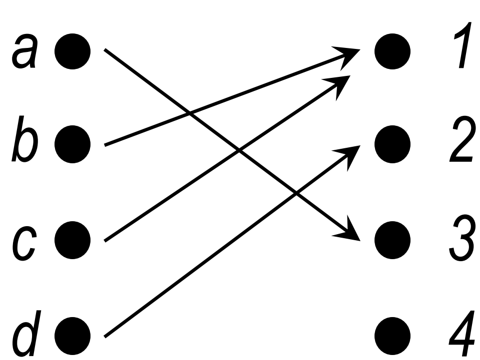
one-to-one, not onto



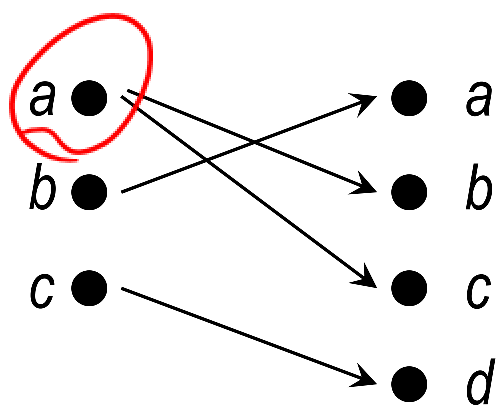
onto, not one-to-one



one-to-one and onto



neither one-to-one nor onto

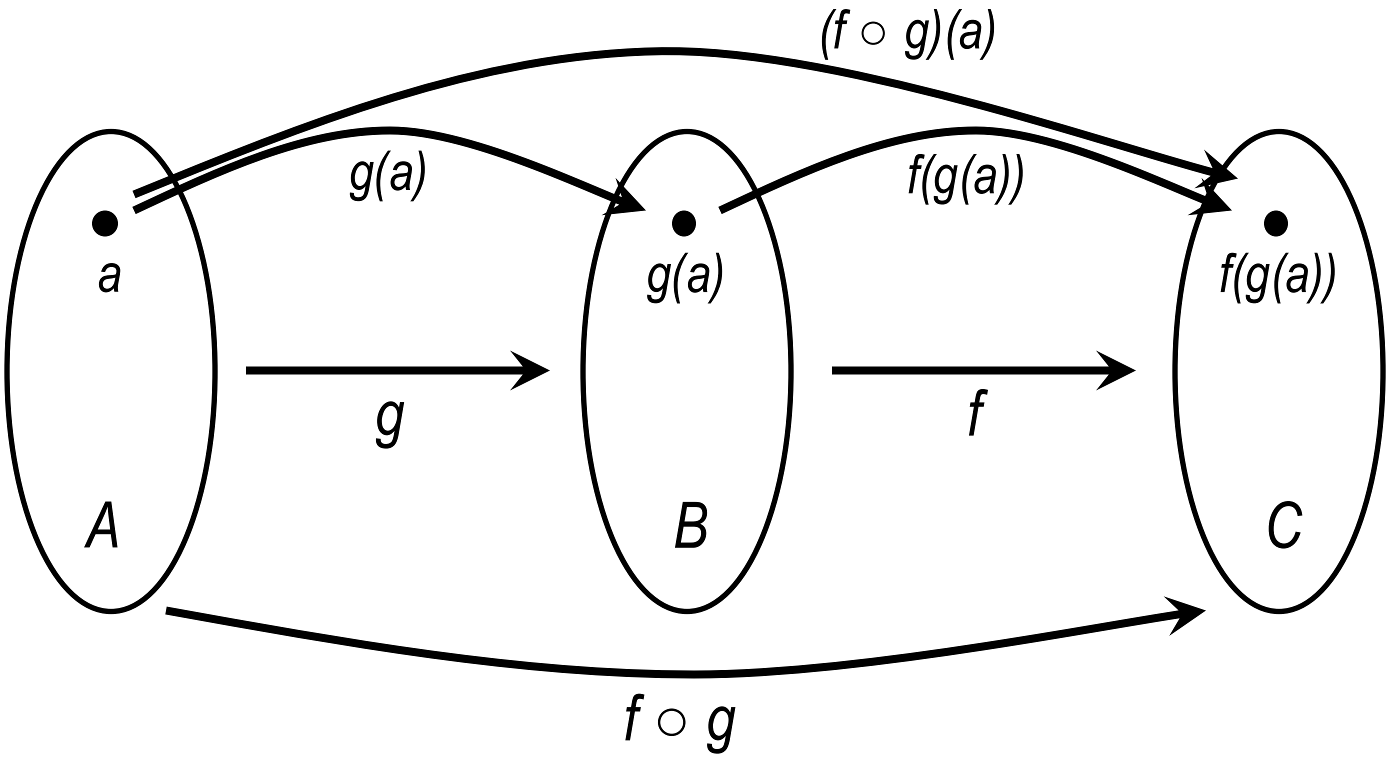


not a function

Composition of Functions

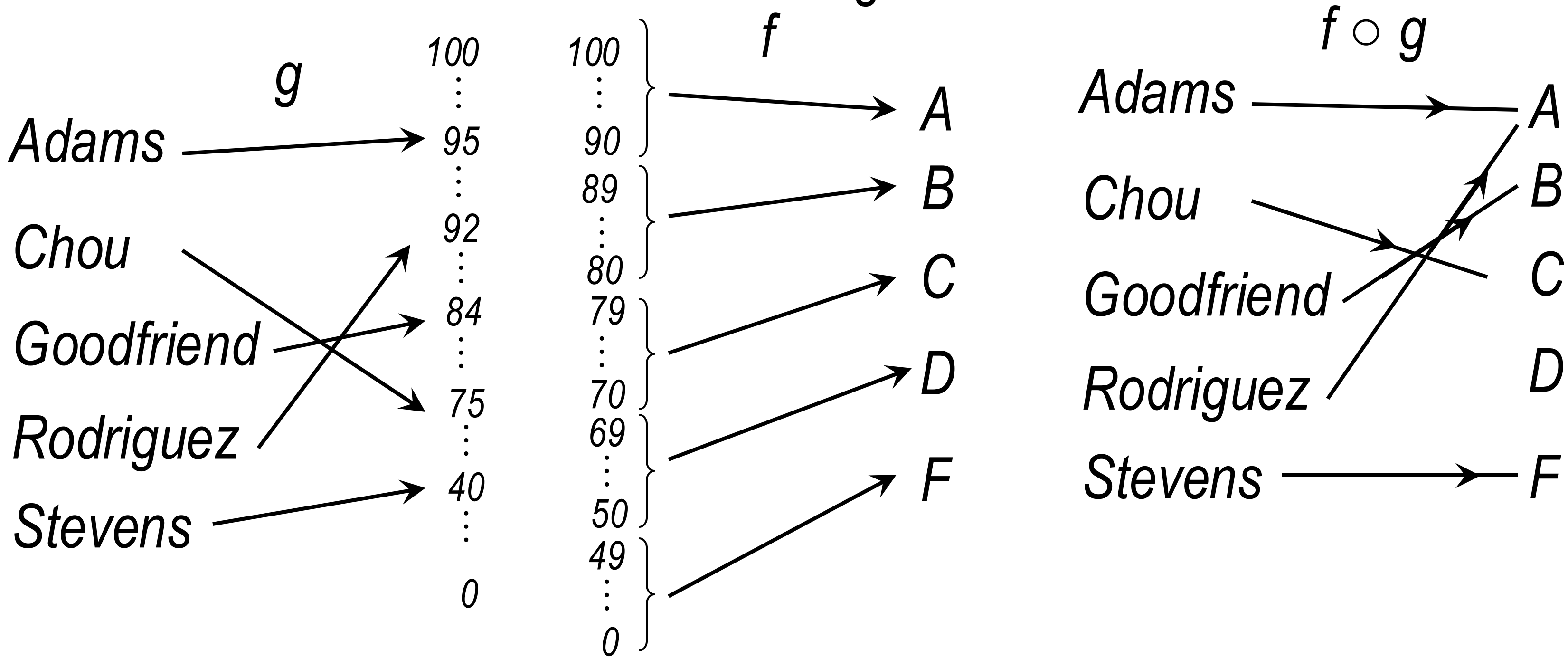
- *Let g be a function from A to B and let f be a function from B to C. The* **composition** *of the functions f and g, denoted by $f \circ g$, is the function from A to C defined by*

$$(f \circ g)(a) = f(g(a))$$



Composition of Functions (cntd)

- *Suppose that the students first get numerical grades from 0 to 100 that are later converted into letter grade.*



- *Let $f(a) = b$ mean `b is the father of a'.*

  What is $f \circ f$?

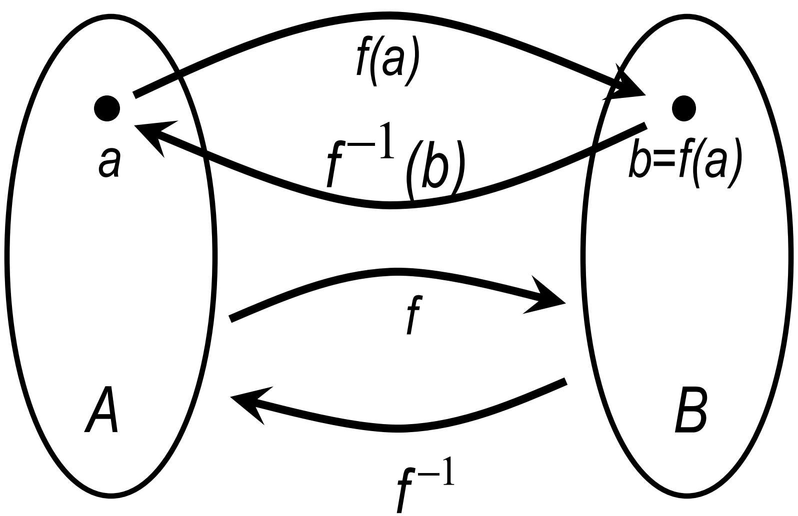
Composition of Numerical Functions

- *Let* $g(x) = x^2$ *and* $f(x) = x + 1$. *Then*
  $(f \circ g)(x) = f(g(x)) = g(x) + 1 = x^2 + 1$

- *Thus, to find the composition of numerical functions f and g given by formulas we have to substitute g(x) instead of x in f(x).*

Inverse Functions

- *Let f be a bijection from the set A to the set B.*
- *The* **inverse** *of f, denoted f^{-1}, is the function that assigns to an element $b \in B$ the unique preimage $a \in A$ such that $f(a) = b$.*
- *Logically, $f^{-1}(b) = a$ if and only if $f(a) = b$*



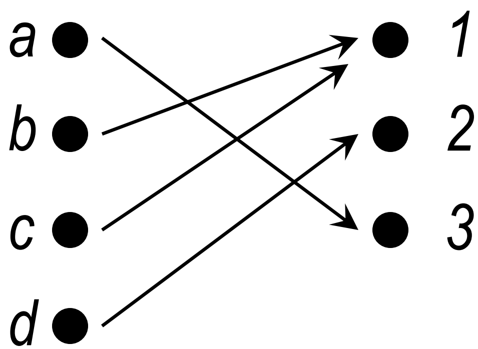
Note!
f^{-1} *does not mean* $\frac{1}{f(x)}$

$$f \circ f^{-1} = i_B$$

$$f^{-1} \circ f = i_A$$

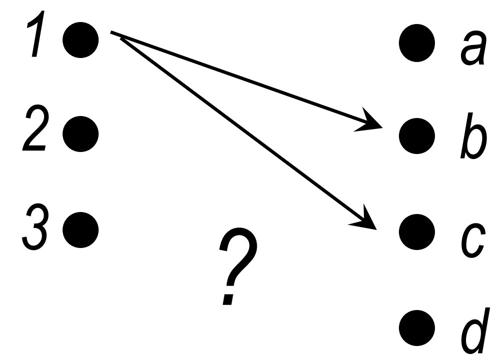
Inverse Functions (cntd)

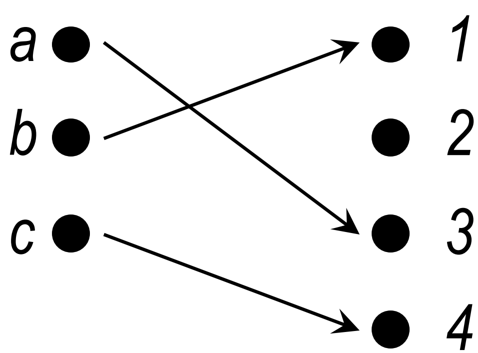
- *If a function f is not a bijection, the inverse function does not exist. Why?*
- *If f is not a bijection, it is either not one-to-one, or not onto*



onto, not one-to-one

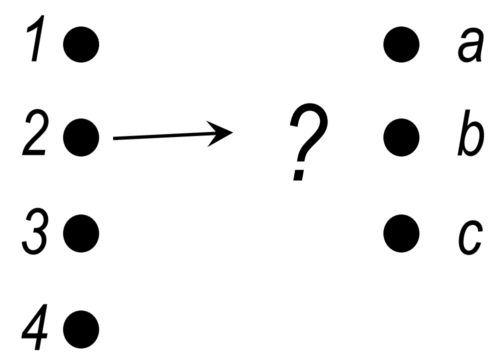
$f^{-1}(1) = ?$





one-to-one, not onto

$f^{-1}(2) = ?$



Practice

Exercises from the Book:

7th edition: 2, 3, 5, 10, 11, 20a, 33a, 36 (page 152 – 154)

8th edition: 2, 3, 5, 10, 11, 20a, 33a, 38 (page 161 – 163)

- *Show that the relation R on sets defined as*

  $(A,B) \in R$ if and only if there exists a bijection $f: A \to B$

  is an equivalence relation