# Cardinality

*Discrete Mathematics*

*Courtesy of Andrei Bulatov*

# Properties of Functions

- *A function $f$ is said to be* one-to-one, *or* injective, *if and only if $f(a) = f(b)$ implies $a = b$.*

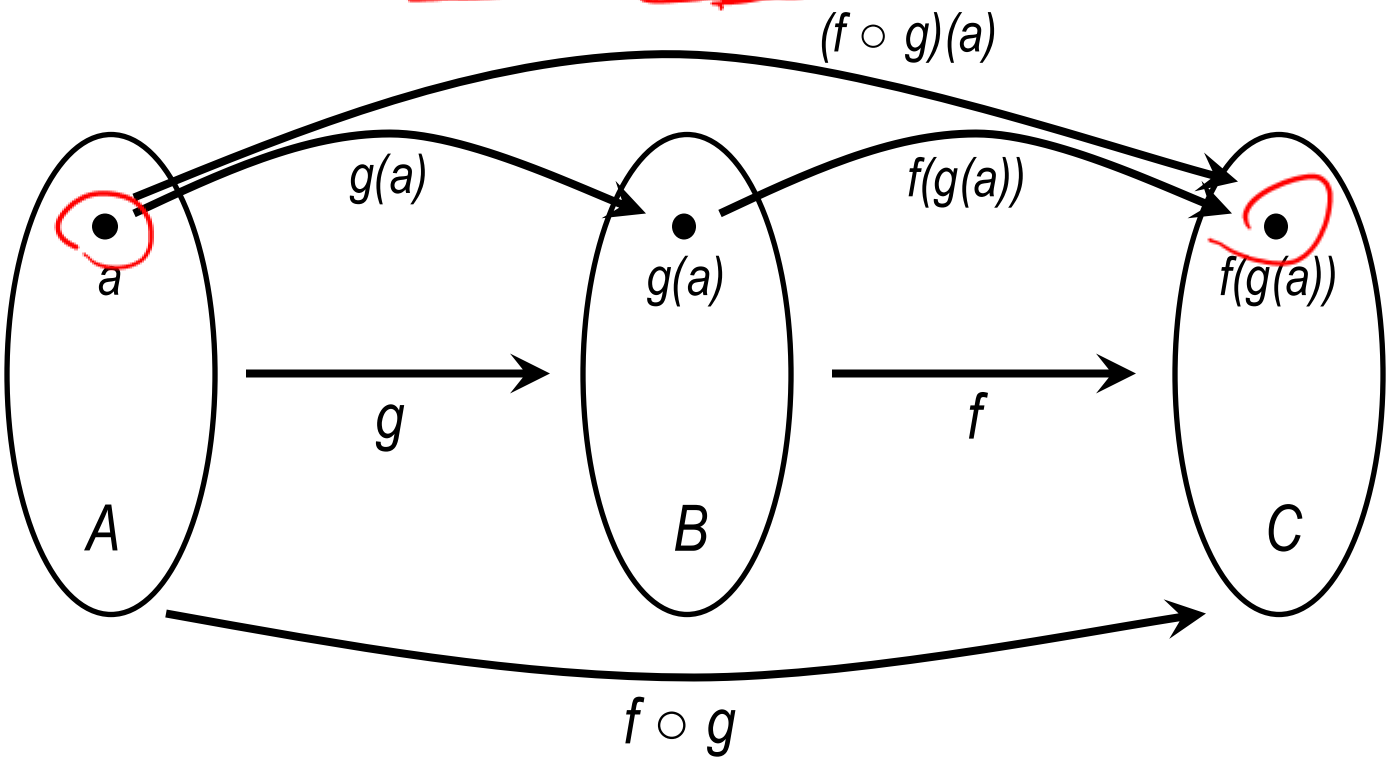
- *A function $f$ from $A$ to $B$ is called* onto, *or* surjective, *if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function is called a* surjection *if it is onto.*

- *A function $f$ is a* one-to-one correspondence, *or a* bijection, *if it is both one-to-one and onto.*

# Composition of Functions

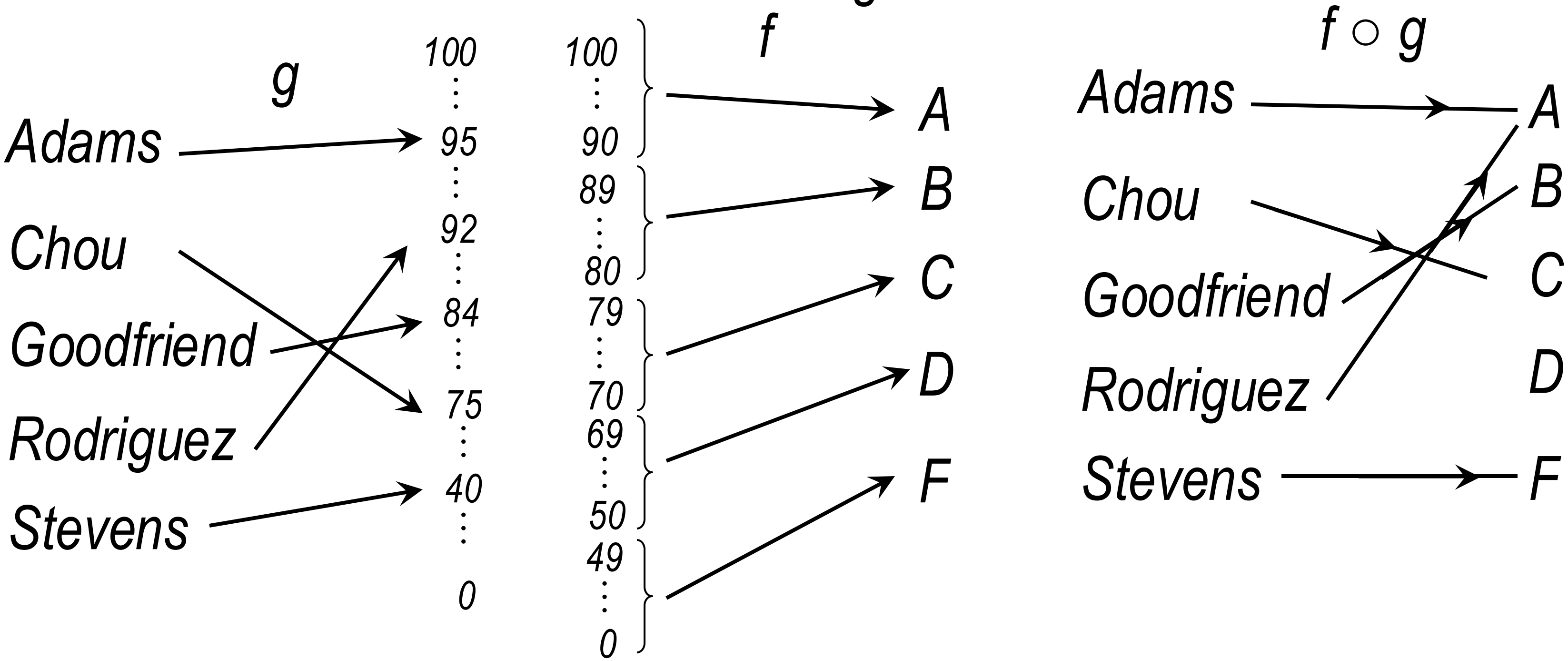
- *Let g be a function from A to B and let f be a function from B to C. The* **composition** *of the functions f and g, denoted by $f \circ g$, is the function from A to C defined by*

$$(f \circ g)(a) = f(g(a))$$



# Composition of Functions (cntd)

- *Suppose that the students first get numerical grades from 0 to 100 that are later converted into letter grade.*




- *Let $f(a) = b$ mean `$b$ is the father of $a$'.*

  *What is $f \circ f$ ?*

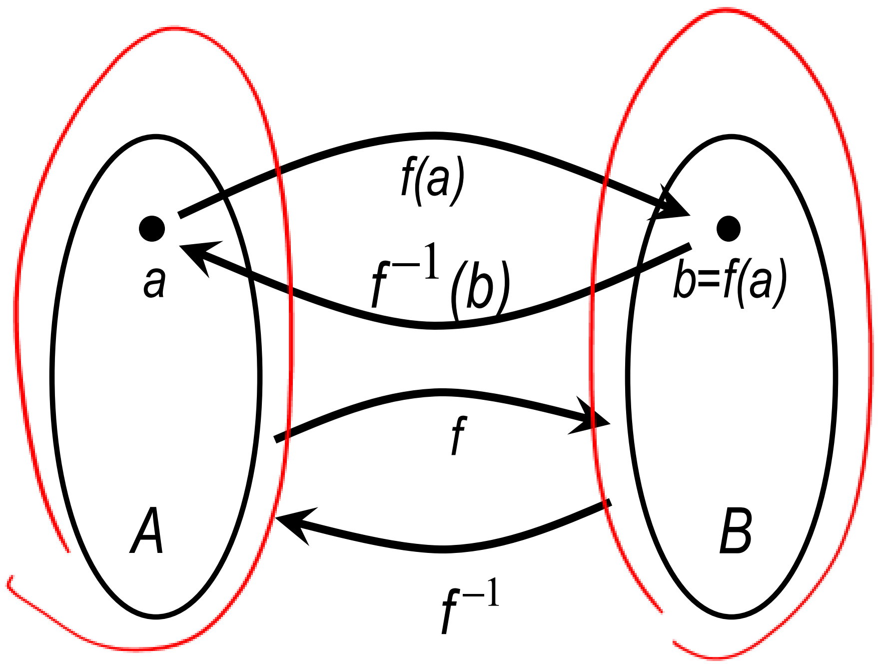
# Composition of Numerical Functions

- *Let* $g(x) = x^2$ *and* $f(x) = x + 1$. *Then*
  $(f \circ g)(x) = f(g(x)) = g(x) + 1 = x^2 + 1$

- *Thus, to find the composition of numerical functions f and g given by formulas we have to substitute x with g(x) in f(x).*

# Inverse Functions

- *Let f be a bijection from the set A to the set B.*
- *The* **inverse** *of f, denoted $f^{-1}$, is the function that assigns to an element $b \in B$ the unique preimage $a \in A$ such that $f(a) = b$.*
- *Logically, $f^{-1}(b) = a$ if and only if $f(a) = b$*



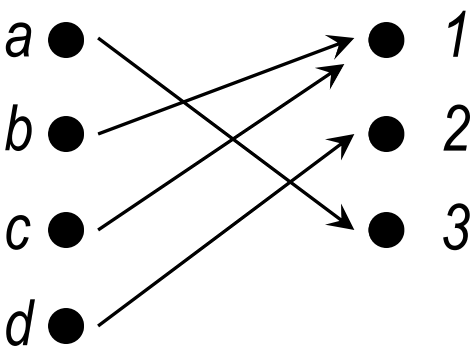

*Note!*
$f^{-1}$ *does not mean* $\frac{1}{f(x)}$

$f \circ f^{-1} = i_B$

$f^{-1} \circ f = i_A$

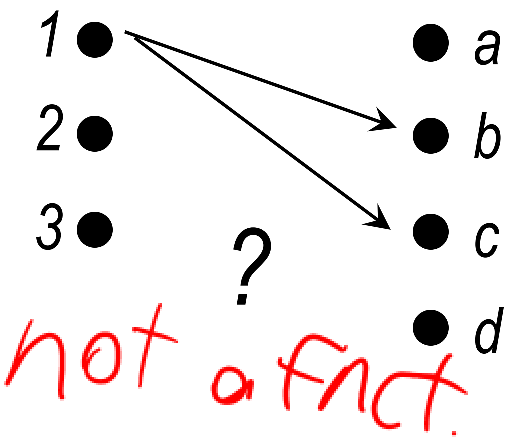
# Inverse Functions (cntd)

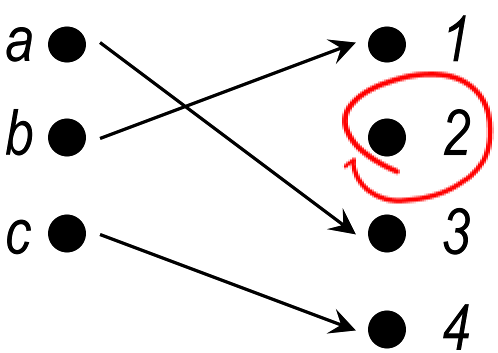
- *If a function $f$ is not a bijection, the inverse function does not exist. Why?*
- *If $f$ is not a bijection, it is either not one-to-one, or not onto*



*onto, not one-to-one*

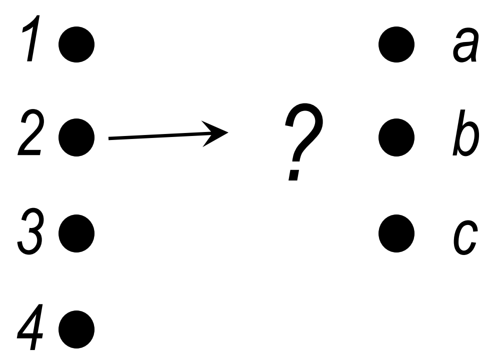
$$f^{-1}(1) = ?$$



*one-to-one, not onto*

$$f^{-1}(2) = ?$$



# How to Count Elements in a Set

- *How many elements are in a set?*
- *Easy for finite sets, just count the elements.*
- *What about infinite sets? Does it make sense at all to ask about the number of elements in an infinite set?*
- *Can we say that this infinite set is larger than that infinite set?*
- *Which set is larger:*

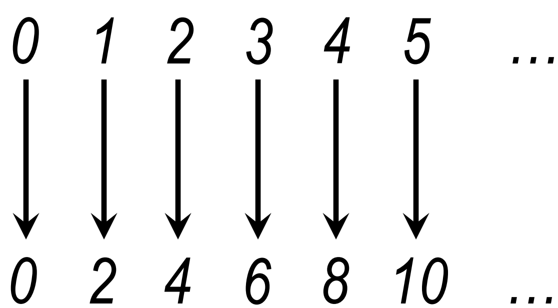
  *- the set of all integers or the set of even integers?* same size

  *- the set of all integers or the set of all of all rationals?* same size

  *- the set of all integers or the set of all reals?* $|\mathbb{R}| > |\mathbb{Z}|$

# Cardinality and Bijections

- *Sets A and B (finite or infinite) have the same cardinality if and only if there is a bijection from A to B*
- $|\mathbb{N}| = |2\mathbb{N}|$



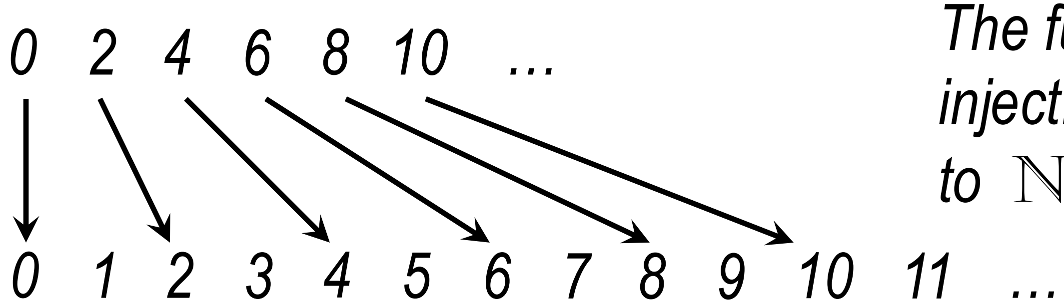

*The function* $f: \mathbb{N} \to 2\mathbb{N}$, *where*
$f(x) = 2x$,
*is a bijection*

$2x = 2y \Rightarrow x = y \quad \therefore$ 1-1

$b = 2a \Rightarrow f(a) = b \quad \therefore$ onto

# Comparing Cardinalities

- Let $A$ and $B$ be sets. We say that $|A| \leq |B|$ if there is an injective function from $A$ to $B$.



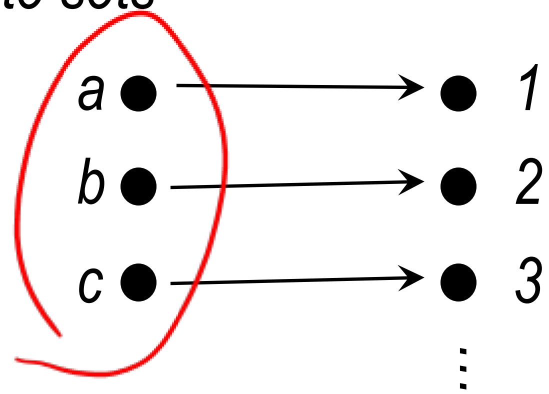

The function $f(x) = x$ is an injective function from $2\mathbb{N}$ to $\mathbb{N}$. Therefore $|2\mathbb{N}| \leq |\mathbb{N}|$

- If there is an injective function from $A$ to $B$, but not from $B$ to $A$, we say that $|A| < |B|$
- If there is an injective function from $A$ to $B$ and an injective function from $B$ to $A$, then we say that $A$ and $B$ have the same cardinality
- Exercise: Prove that a bijection from $A$ to $B$ exists if and only if there are injective functions from $A$ to $B$ and from $B$ to $A$.

# Countable and Uncountable

- *A set A is said to be* **countable** *if* $|A| \le |\mathbb{N}|$
- *This is because an injective function from A to* $\mathbb{N}$ *can be viewed as assigning numbers to the elements of A, thus counting them*
- *Sets that are not countable are called* **uncountable**
- *Countable sets:*

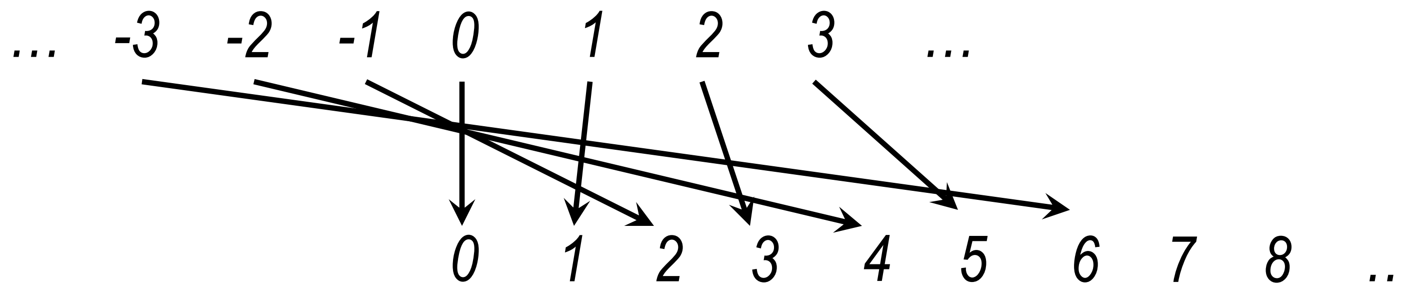
  *finite sets*




  *any subset of* $\mathbb{N}$

# More Countable Sets

- *The set of all integers is countable*




- *In other words we can make a list of all integers*

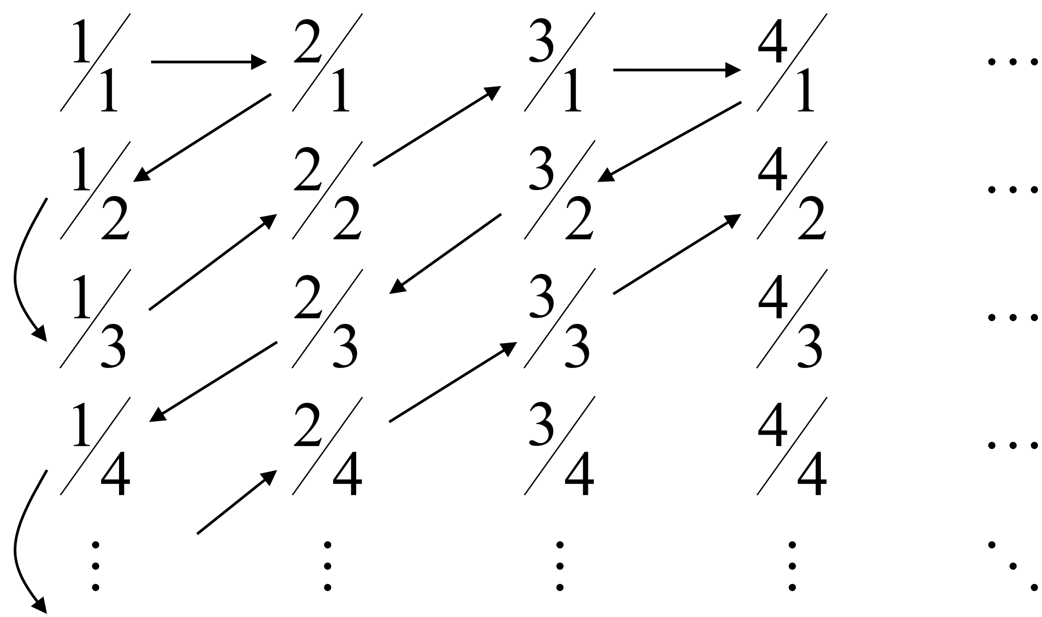
*0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, …*

- *The cardinality of the set of all natural numbers is denoted by* $\aleph_0$

*"aleph-zero"*

# More Countable Sets (cntd)

- *The set of positive rational numbers is countable*
- *Every rational number can be represented as a fraction* $p/q$
  *We do not insist that* $p$ *and* $q$ *do not have a common divisor*




- *This gives an injection from* $\mathbb{Q}^+$ *to* $\mathbb{N}$*. The converse injection is* $f(x) = x + 1$

# The Smallest Infinite Set

- *Theorem.*

  *If $A$ is an infinite set, then* $|A| \geq \aleph_0$

- *Proof requires mathematical induction. Wait for a few days.*

# Uncountable Sets

- *Can we make a list of all real numbers?*
- *Every real number can be represented as an infinite decimal fraction, like 3.14159265358979323846264338327950288841971…*
- *Suppose we have constructed a list of all real numbers*

1. $a_{10}.a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}\dots$
2. $a_{20}.a_{21}a_{22}a_{23}a_{24}a_{25}a_{26}a_{27}\dots$
3. $a_{30}.a_{31}a_{32}a_{33}a_{34}a_{35}a_{36}a_{37}\dots$
4. $a_{40}.a_{41}a_{42}a_{43}a_{44}a_{45}a_{46}a_{47}\dots$
5. $a_{50}.a_{51}a_{52}a_{53}a_{54}a_{55}a_{56}a_{57}\dots$

$\vdots \qquad \vdots$

*Here the* $a_{ij}$ *are digits 0,1,2,…,9*

*Let*

$$b_i = \begin{cases} 4, & \text{if } a_{ii} \neq 4, \\ 5 & \text{otherwise} \end{cases}$$

- *It is not hard to see that the number* $0.b_1b_2b_3b_4b_5b_6b_7\dots$ *is not in this list*

# Cantor's Theorem

- **Theorem** *(Cantor). For any set* $|P(A)| > |A|$.

  **Proof**.

  *It is easy to see that* $|P(A)| \geq |A|$. *(Find an injection* $A \to P(A)$ *)*

  *Prove* $|P(A)| \neq |A|$. *Suppose there is a bijection* $f\colon A \to P(A)$.

  *We find a set that does not belong to the range of* $f$. *A contradiction with the assumption that* $f$ *is bijective.*

  *Consider the set* $T = \{ a \in A \mid a \notin f(a) \}$

  *If* $T$ *is in the range of* $f$, *then there is* $t \in A$ *such that* $f(t) = T$.

  *Either* $t \in T$ *or* $t \notin T$.

  *If* $t \in T$ *then* $t \in f(t)$, *and we get* $t \notin T$.

  *If* $t \notin T$ *then* $t \in T$.

  *Q.E.D.*

# Cantor's Theorem (cntd)

- *This method is called Cantor's diagonalization method*
- *The cardinality of P(A) is denoted by* $2^{|A|}$
- *Thus, we obtain an infinite series of infinite cardinals*

$$|\mathbb{N}| = \aleph_0$$

$$2^{\aleph_0} = \aleph_1 \quad (= |\mathbb{R}|)$$

$$2^{\aleph_1} = \aleph_2$$

$$\vdots$$

# Continuum Hypothesis

- *We just proved that* $\aleph_0 < |\mathbb{R}|$. *Does there exist a set* $A$ *such that* $\aleph_0 < |A| < |\mathbb{R}|$?

- *The negative answer to this question is known as the continuum hypothesis.*

- *Continuum hypothesis is the first problem in the list of Hilbert's problems*

- *Paul Cohen resolved the question in 1963.*

  *...It can't be proven or disproven in ZFC.*

# Practice

- *Determine whether each of the following statements is true:*
  - *If $A, B$ are countable sets then $A \cup B$ is countable*
  - *If $A, B$ are uncountable sets then $A \cap B$ is uncountable*
  - *If $A, B$ are countable sets then $A - B$ is countable*

- *Find a one-to-one correspondence between $\mathbb{Z}^+$ and $\{2,6,10,14,\ldots\}$*

- *Let $I = \{r \in \mathbb{R} \mid r$ is irrational$\} = \mathbb{R} - \mathbb{Q}$. Is $I$ countable or uncountable?*

- *Construct a bijective mapping between the closed interval [0;1] and the square $[0;1] \times [0;1]$*