Mathematical Induction

Discrete Mathematics

Courtesy of Andrei Bulatov

Cardinality & Counting

- *$|A| \leq |B|$ if there is an injective function from A to B*
- *A set A is said to be countable if there is an injective function from A to $\mathbb{N}$*
  - *E.g.* $f(x) = \begin{cases} -2x \ \ if\ x < 0 \\ 2x + 1\ otherwise \end{cases}$ *is an injective function from $\mathbb{Z}$ to $\mathbb{N}$*
- *Otherwise, it is uncountable*

More Countable Sets

- *The set of positive rational numbers is countable*
- *Every rational number can be represented as a fraction* p/q

We do not insist that p *and* q *do not have a common divisor*

$$\begin{array}{ccccc}
1/1 & 2/1 & 3/1 & 4/1 & \cdots \\
1/2 & 2/2 & 3/2 & 4/2 & \cdots \\
1/3 & 2/3 & 3/3 & 4/3 & \cdots \\
1/4 & 2/4 & 3/4 & 4/4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

- *This gives an injection from* $\mathbb{Q}^+$ *to* $\mathbb{N}$. *The converse injection is* $f(x) = x + 1$

Uncountable Sets

- *To show something is uncountable, need to show there does not exist an injective function from A to $\mathbb{N}$*
- *Usually done by contradiction*
- *Are the real numbers countable? Assume yes!*

$$
\begin{array}{ll}
1. & a_{10}.a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}\ldots \\
2. & a_{20}.a_{21}a_{22}a_{23}a_{24}a_{25}a_{26}a_{27}\ldots \\
3. & a_{30}.a_{31}a_{32}a_{33}a_{34}a_{35}a_{36}a_{37}\ldots \\
4. & a_{40}.a_{41}a_{42}a_{43}a_{44}a_{45}a_{46}a_{47}\ldots \\
5. & a_{50}.a_{51}a_{52}a_{53}a_{54}a_{55}a_{56}a_{57}\ldots \\
\vdots & \vdots
\end{array}
$$

Here the a_{ij} are digits 0,1,2,…,9

Let

$$b_i = \begin{cases} 4, & \text{if } a_{ii} \neq 4, \\ 5 & \text{otherwise} \end{cases}$$

- *Is the number $0.b_1b_2b_3b_4b_5b_6b_7\ldots$ in this list?*

K $b_k = \begin{cases} 4 & \text{if } a_{kk} \neq \\ 5 \end{cases}$

Cantor's Theorem

- **Theorem** *(Cantor). For any set* $|P(A)| > |A|$.

  Proof.

  It is easy to see that $|P(A)| \geq |A|$. *(Find an injection* $A \to P(A)$ *)*

  Prove $|P(A)| \neq |A|$. *Suppose there is a bijection* $f\colon A \to P(A)$.

Consider the set $T = \{ a \in A \mid a \notin f(a) \}$. *We will show that T is not in the range of f*

If T *is in the range of* f, *then there is* $t \in A$ *such that* $f(t) = T$.

Either $t \in T$ *or* $t \notin T$.

If $t \in T$ *then* $t \in f(t)$, *and we get* $t \notin T$.

If $t \notin T$ *then* $t \in T$. Q.E.D.

Familiar?

Cantor's Theorem (cntd)

- *This method is called Cantor's diagonalization method*
- *The cardinality of P(A) is denoted by* $2^{|A|}$
- *Thus, we obtain an infinite series of infinite cardinals*

$$
\begin{aligned}
|\mathbb{N}| &= \aleph_0 \\
2^{\aleph_0} &= \aleph_1 \qquad (= |\mathbb{R}|) \\
2^{\aleph_1} &= \aleph_2 \\
&\vdots
\end{aligned}
$$

Continuum Hypothesis

- *We just proved that* $\aleph_0 < |\mathbb{R}|$. *Does there exist a set* A *such that* $\aleph_0 < |A| < |\mathbb{R}|$?

- *The negative answer to this question is known as the continuum hypothesis.*

- *Continuum hypothesis is the first problem in the list of Hilbert's problems*

- *Paul Cohen resolved the question in 1963.*

  ...It can't be proven or disproven in ZFC.

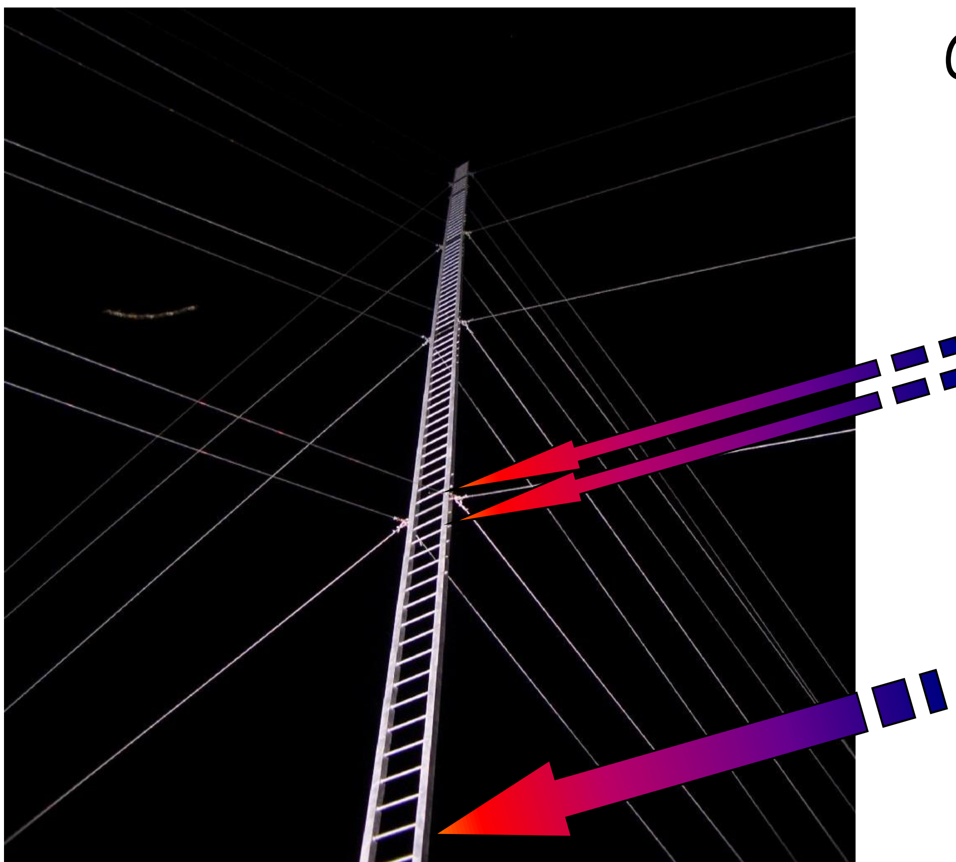
The Smallest Infinite Set

- *What about an (infinite) set smaller than* $\aleph_0$ *?* $\mathbb{N}$
- *Theorem.*

  If A *is an infinite set, then* $|A| \geq \aleph_0$
- *Proof requires mathematical induction...*

Principle of Mathematical Induction

- *Idea is to prove properties about infinite but countable sets by showing that each element implies the next*



Climbing an infinite ladder

We can reach every step of it, if

For all k, standing on the rung k we can step on the rung $k+1$

We can reach the first rung

Principle of Mathematical Induction

- *Principle of mathematical induction*:

$$(P(1) \wedge \forall k\, (P(k) \rightarrow P(k+1))) \rightarrow \forall n \in \mathbb{Z}^{+}\, P(n)$$

- *Proof by induction uses the principle of induction to prove that a property P(n) holds for every positive integer n*

- *Comprises two steps:*

  Basis step: *We show that P(1) is true.*

  Inductive step: *We show that if P(k) is true for some positive integer k, then P(k+1) is also true*

- *Explicitly, for the inductive step, assume P(k) is true for some k (the inductive hypothesis) and show that P(k+1) is also true*

The Domino Effect

- *Goal: Show that all dominos fall if we push the first:*
- *Basis Step: The first domino falls because we pushed it*
- *Inductive step: Assume the kth domino fell (inductive hypothesis). Then the (k+1)th domino will be hit and also fall*

Summation

- *Prove that the sum of the first n natural numbers equals* $\frac{n(n+1)}{2}$ *that is* $$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

- *P(n): `the sum of the first n natural numbers equals* $\frac{n(n+1)}{2}$
- *Basis step: P(1) means* $1=\frac{1(1+1)}{2}$ $= 1$

- *Inductive step: Assume the inductive hypothesis, P(k) is true, i.e.*

k+2

$$1+2+3+\cdots+k=\frac{k(k+1)}{2}$$

Prove P(k + 1): $1+2+3+\cdots+k+(k+1)=\frac{(k+1)((k+1)+1)}{2}$

$1+2+3+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$ *By inductive hyp.*

$$=\frac{k(k+1)+2(k+1)}{2}=\frac{(k+1)(k+2)}{2}$$

More Summation

- *Prove that* $1+2+2^2+\cdots+2^n = 2^{n+1}-1$
- *Let P(n) be the statement* `$1+2+2^2+\cdots+2^n = 2^{n+1}-1$' *for the integer n*

- *Basis step: P(0) is true, as* $2^0 = 1 = 2^{0+1}-1$
- *Inductive step: We assume the inductive hypothesis*

$$\underline{1+2+2^2+\cdots+2^k} = 2^{k+1}-1$$

and prove P(k + 1), that is

$$\underline{1+2+2^2+\cdots+2^k+2^{k+1}} = 2^{(k+1)+1}-1 = \underline{2^{k+2}-1}$$

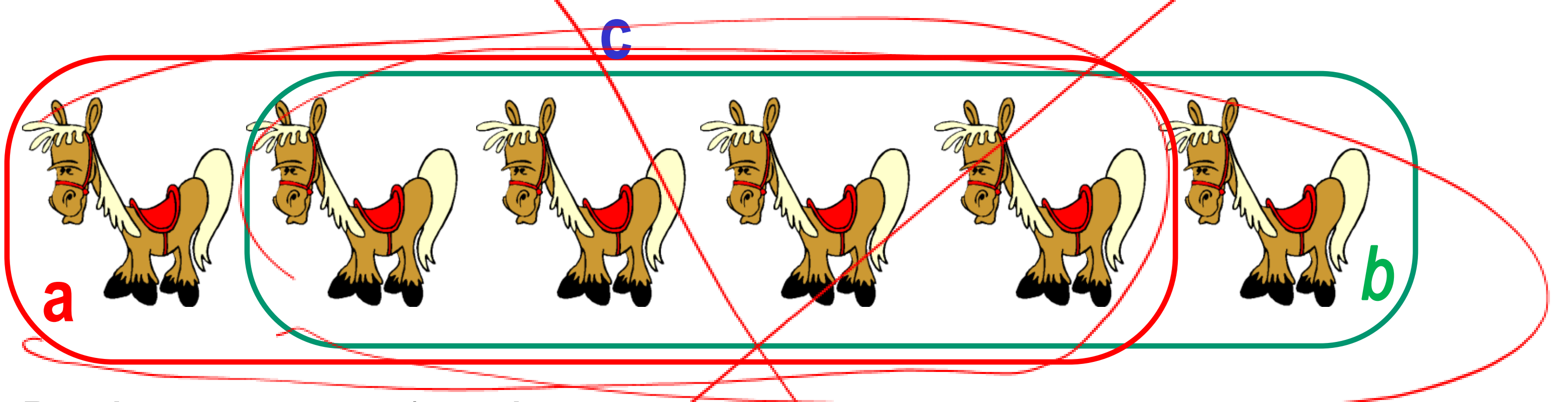
We have $\underbrace{1+2+2^2+\cdots+2^k}+2^{k+1} = (2^{k+1}-1)+2^{k+1}$

$$= 2\cdot 2^{k+1}-1$$

$$= 2^{k+2}-1$$

No Horse of Different Color

- *Prove that in any herd of n horses all horses are of the same color*



- *Basis step: $n = 1$, clear*
- *Inductive step: Inductive hypothesis: Suppose in any herd of k horses all horses have the same color.*
- *Take a herd of $k + 1$ horses, and pick horses a and b there*

  Take two `subherds' containing k horses each such that one contains a, but not b, and the other contains b, but not a

  The intersection contains some horse c. By inductive hypothesis a has the same color as c and b has the same color as c

The Cardinality of the Power Set

- *We have proved that, for any finite set A, it is true that* $|P(A)| = 2^{|A|}$
- *Let Q(n) denote the statement `an n-element set has* 2^n *subsets'*
- *Basis step: Q(1), then P(A) = {{}, A} which has size 2*
- *Inductive step. We make the inductive hypothesis, a k-element set A has* 2^k *subsets*

  We have to prove Q(k + 1), that is if a set A contains k + 1 elements, then $|P(A)| = 2^{k+1}$

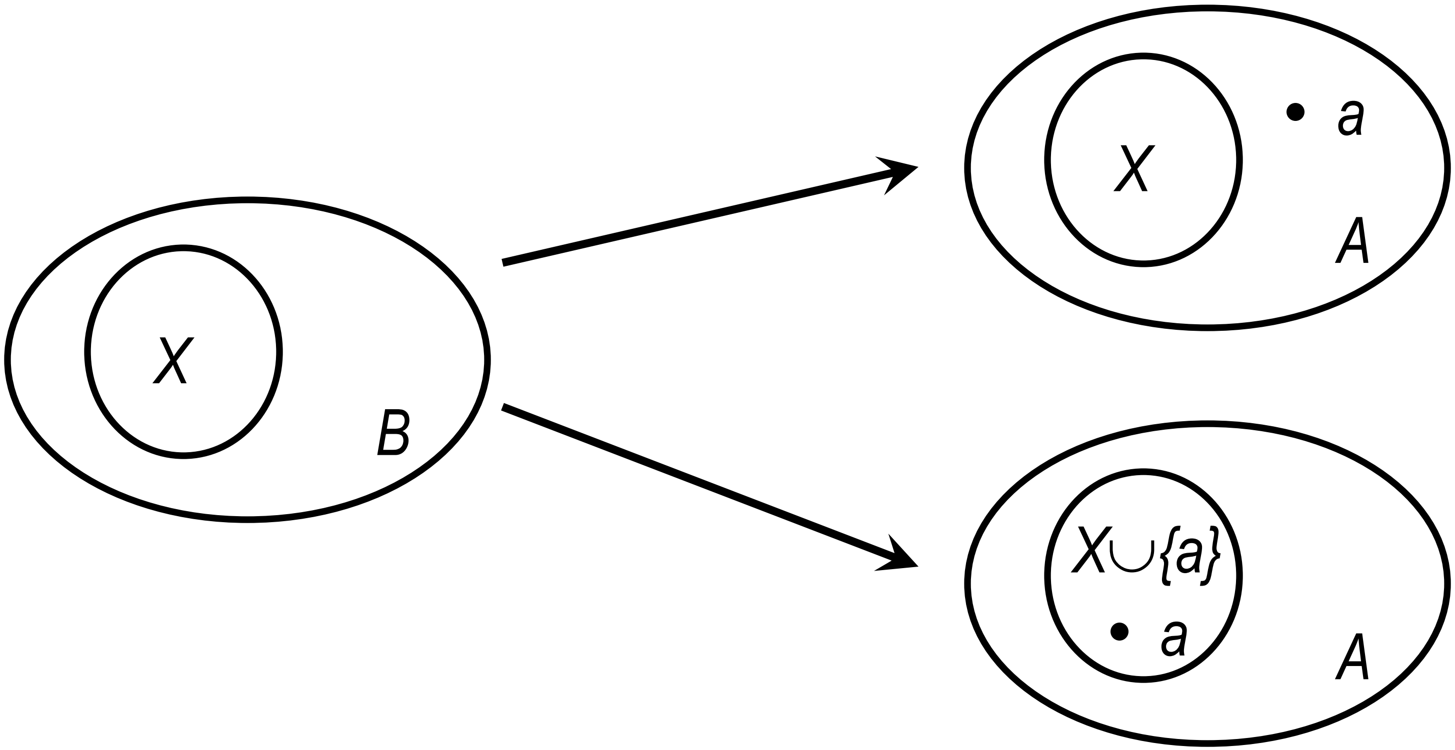
  Fix an element $a \in A$, *and set* $B = A - \{a\}$.

  The set B contains k elements, hence $|P(B)| = 2^k$

  Every subset X of B corresponds to two subsets of A:

  $$X \text{ and } X \cup \{a\}$$

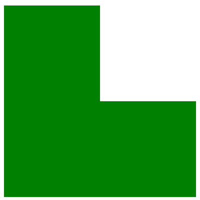
The Cardinality of the Power Set (cntd)



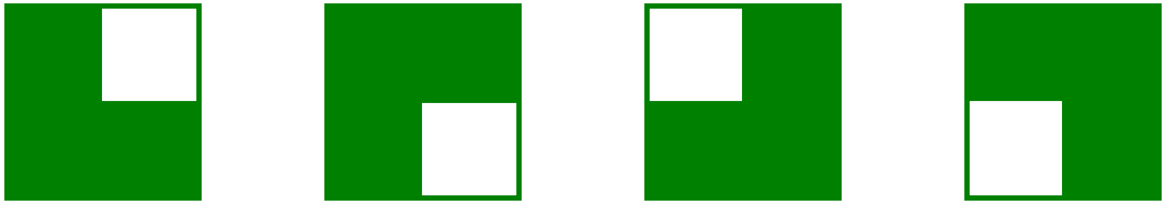
Therefore, $|P(A)| = 2 \cdot |P(B)| = 2 \cdot 2^k = 2^{k+1}$

Triomino

- *Let n be a positive integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using triominos*

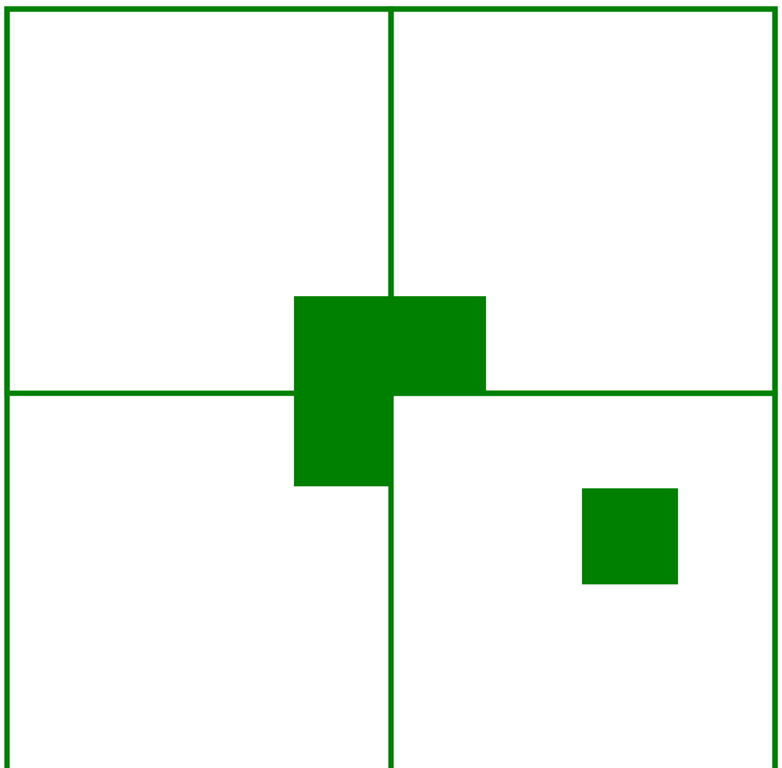


- *$P(n)$ denotes the statement above*
- *Basis step: $P(1)$ is true, as 2×2 checkerboards with one square removed have one of the following shapes*



Triomino (cntd)

- *Inductive step: Suppose that P(k) is true that is every* $2^k \times 2^k$ *checkerboard with one square removed can be tiled with triominos.*
  We have to prove P(k + 1), that is, every $2^{k+1} \times 2^{k+1}$ *checkerboard without one square can be tiled.*



Split the big checkerboard into 4 half-size checkerboards

WLOG assume the bottom right is missing a square. Put one triomino as shown.

We have 4 $2^k \times 2^k$ *checkerboards, each without one square. By the induction hypothesis, they can be tiled.*

Practice

Exercises from the Book:

7th edition: 5,14, 19, 31 (page 329 – 330)

8th edition: 5, 14, 19, 31 (page 350 – 351)