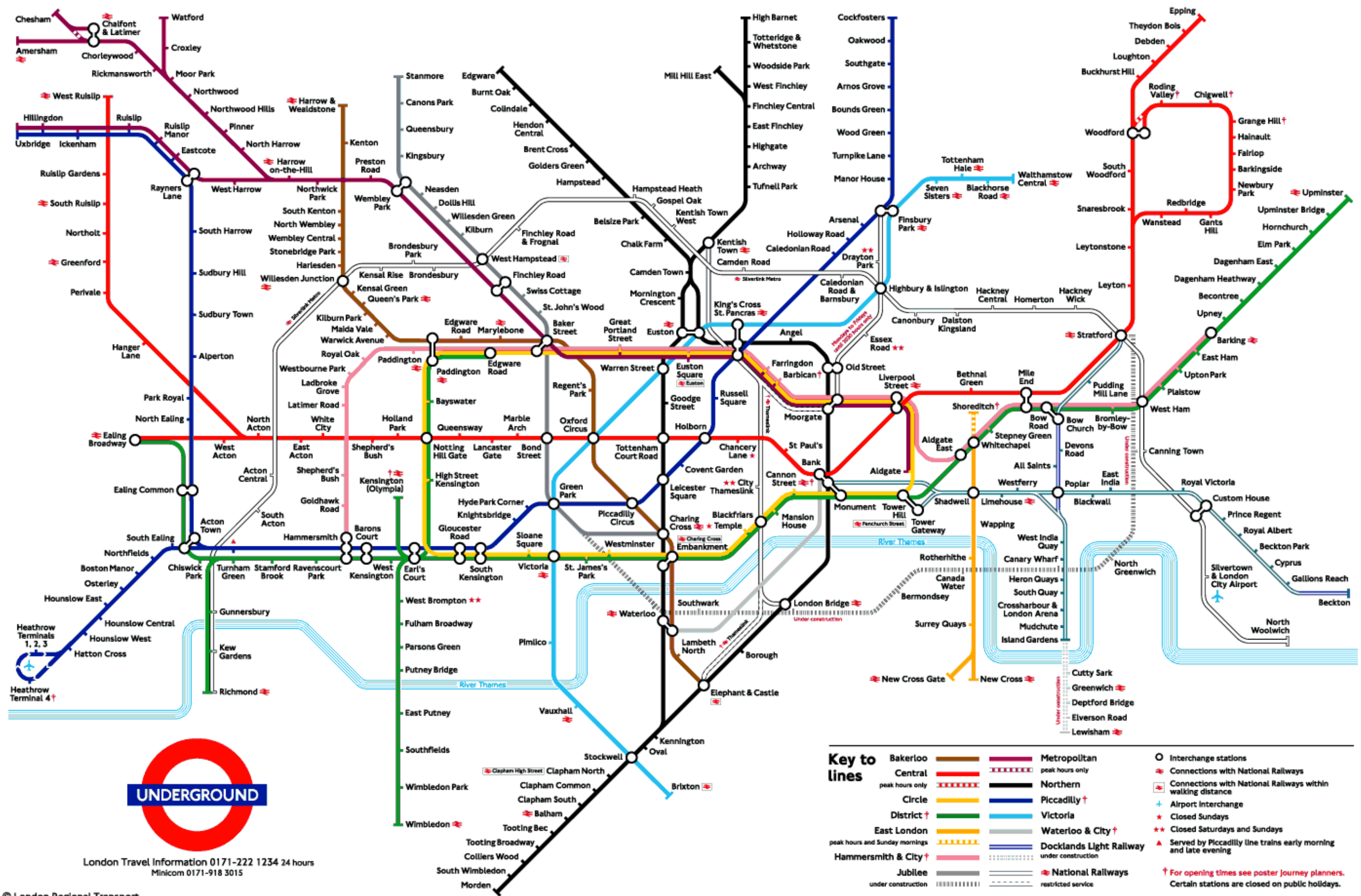
Graphs

Discrete Mathematics

Courtesy of Andrei Bulatov

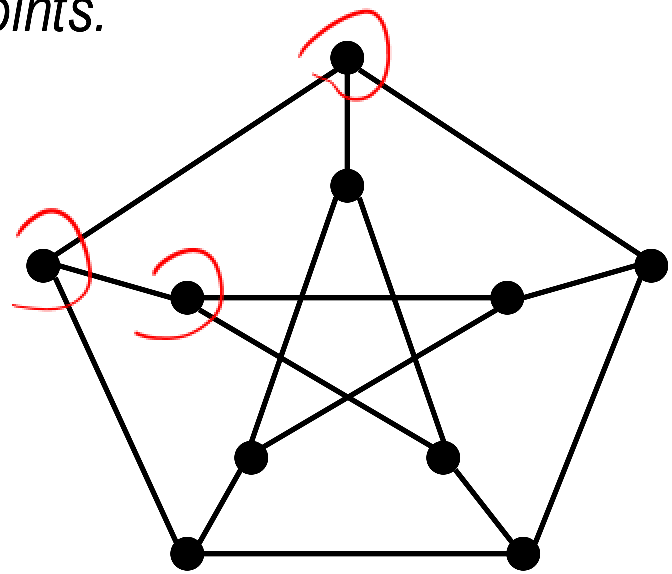
Graphs

- *Often we are interested in relations/connections between objects rather than the nature of these objects and connections*



Graphs (formal definition)

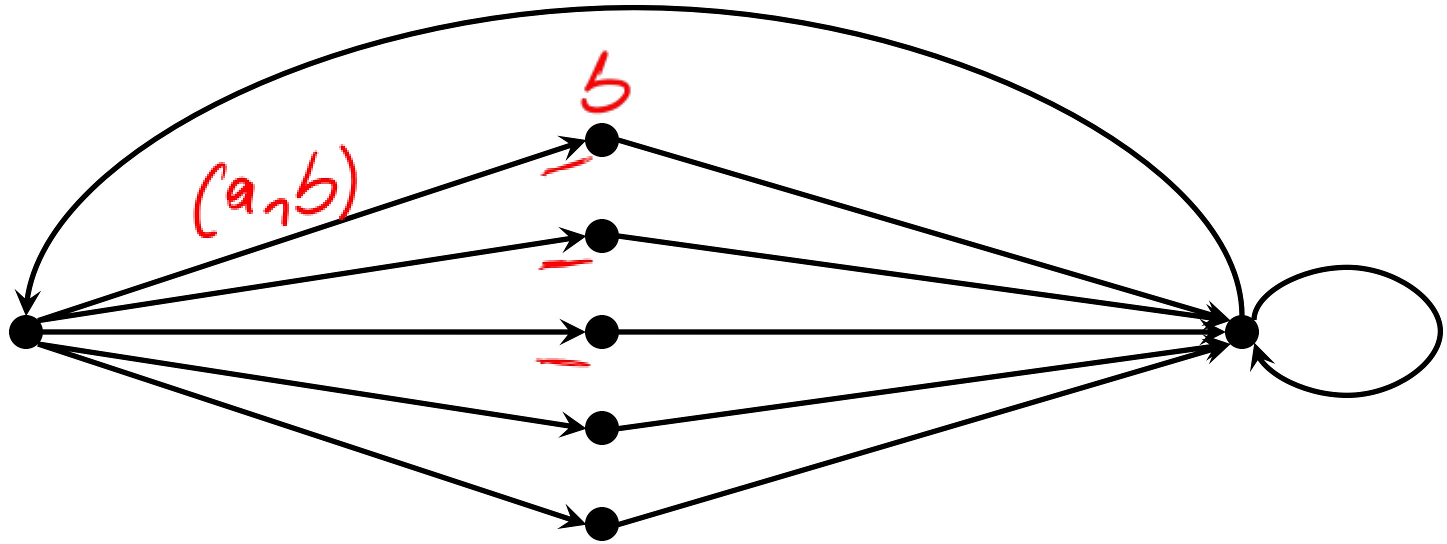
- *A* graph *G = (V, E) consists of V, a nonempty set of* vertices *(or* nodes*) and E, a set of* edges *(or* arcs*). Each edge is an unordered pair ~~or~~ of vertices, called its* endpoints*. An edge is said to connect its endpoints.*



- *Such a graph is also called* undirected

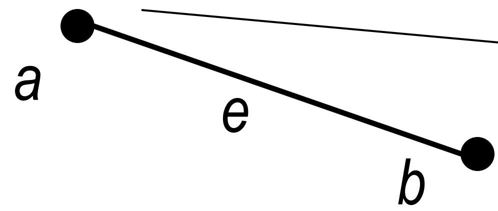
Directed graphs

- *Let V be a nonempty set and $E \subseteq V \times V$. The pair $G = (V, E)$ is called a directed graph, or digraph*



- *Thus, a digraph is just a binary relation on the set V.*
- *An undirected graph is a symmetric binary relation on V.*

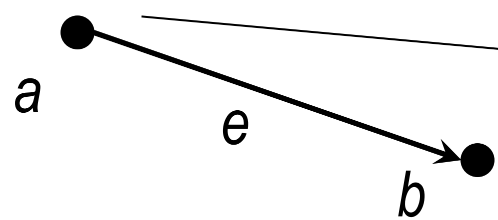
Vertices and Edges

- 

  these vertices are ***adjacent***

  a and b are the ***endpoints*** *of the edge e*

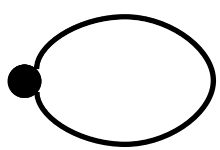
  a and b are ***incident*** *to the edge e*

- 

  a is ***adjacent*** *to b*

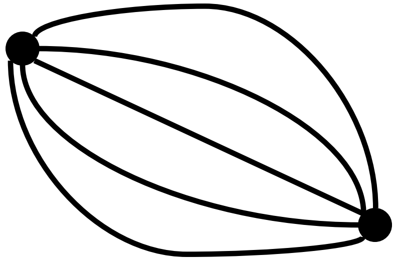
  a is the ***origin*** *(**source**) of the arc e*

  b is the ***terminal vertex*** *of the arc e*

- 

  this edge is a ***loop***

Vertices and Edges (cntd)

- 

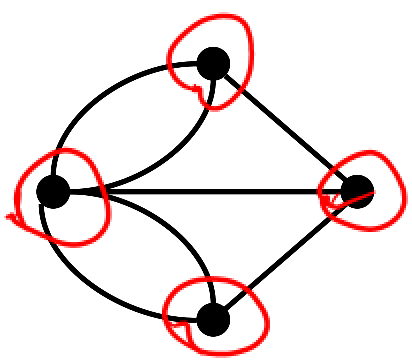
  these two vertices are connected with multiple edges

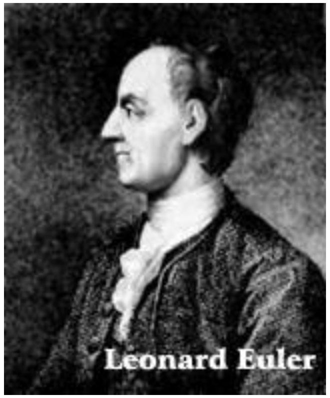
- *A graph with multiple edges is called a multigraph*

- *A graph (digraph) without loops and multiple edges is called an simple (di)graph*

The Seven Bridges of Königsberg

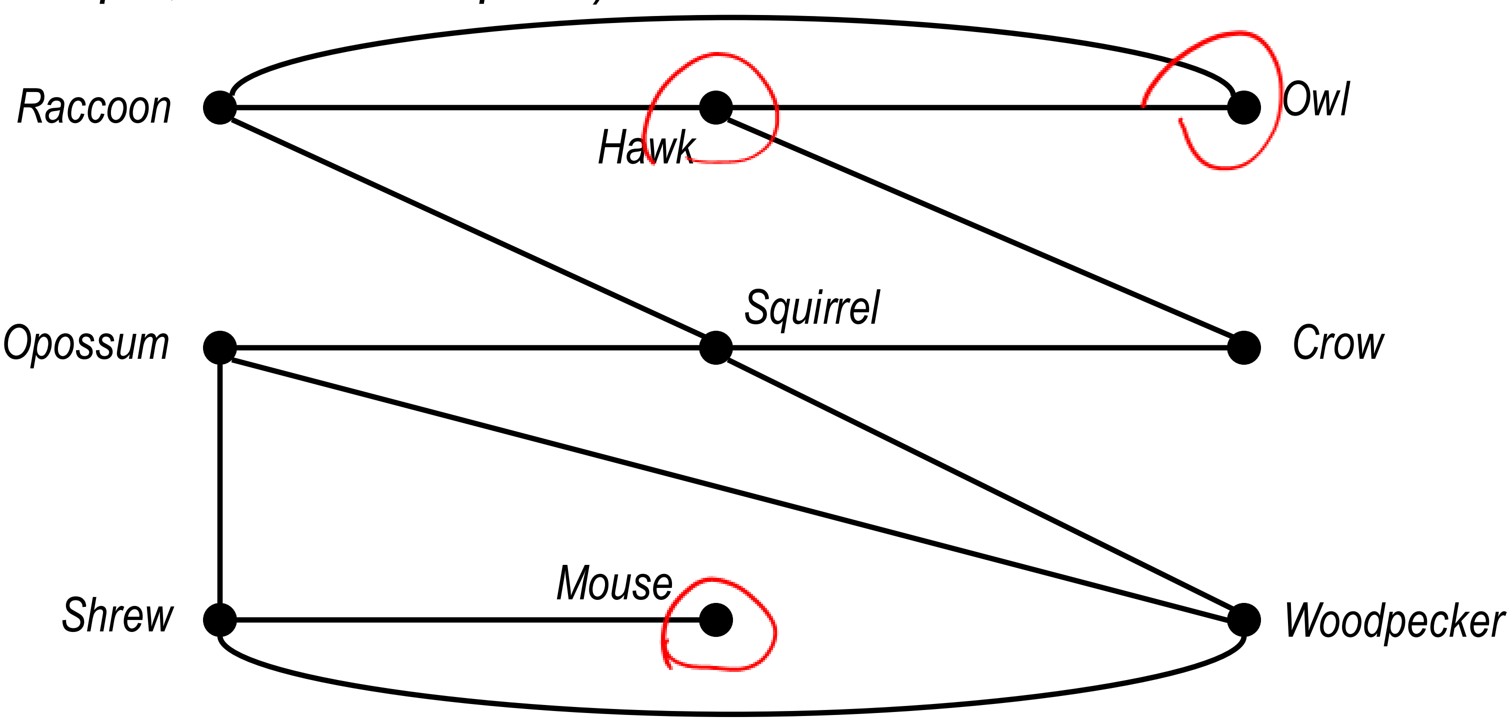
- In the downtown of Konigsberg (previously in Prussia) there are seven bridges across the river Pregel and islands
- Question: Is there a walk that uses every bridge exactly once and returns to the origin
- Euler solved this problem in 1736; starting graph theory





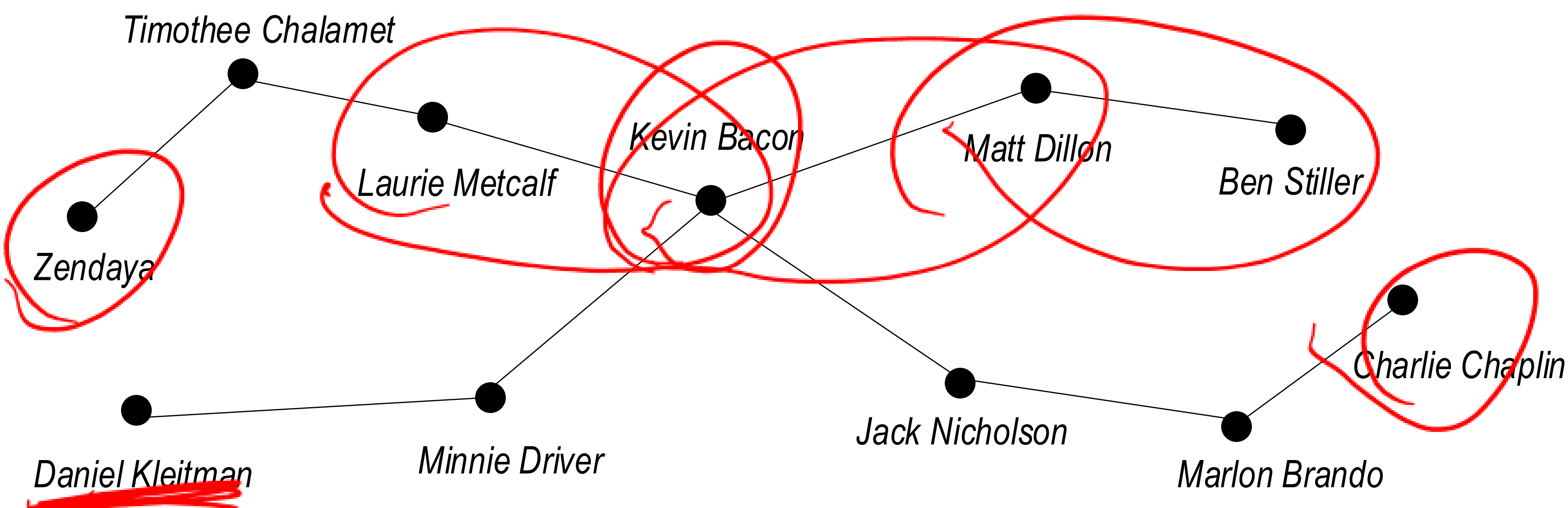
Applications: Niche Overlap Graph

- *The competition between species in an ecosystem can be modeled using a niche overlap graph.*
- *Each species is represented by a vertex. An edge connects two vertices if the two species represented by these vertices compete (for example, for food or space).*



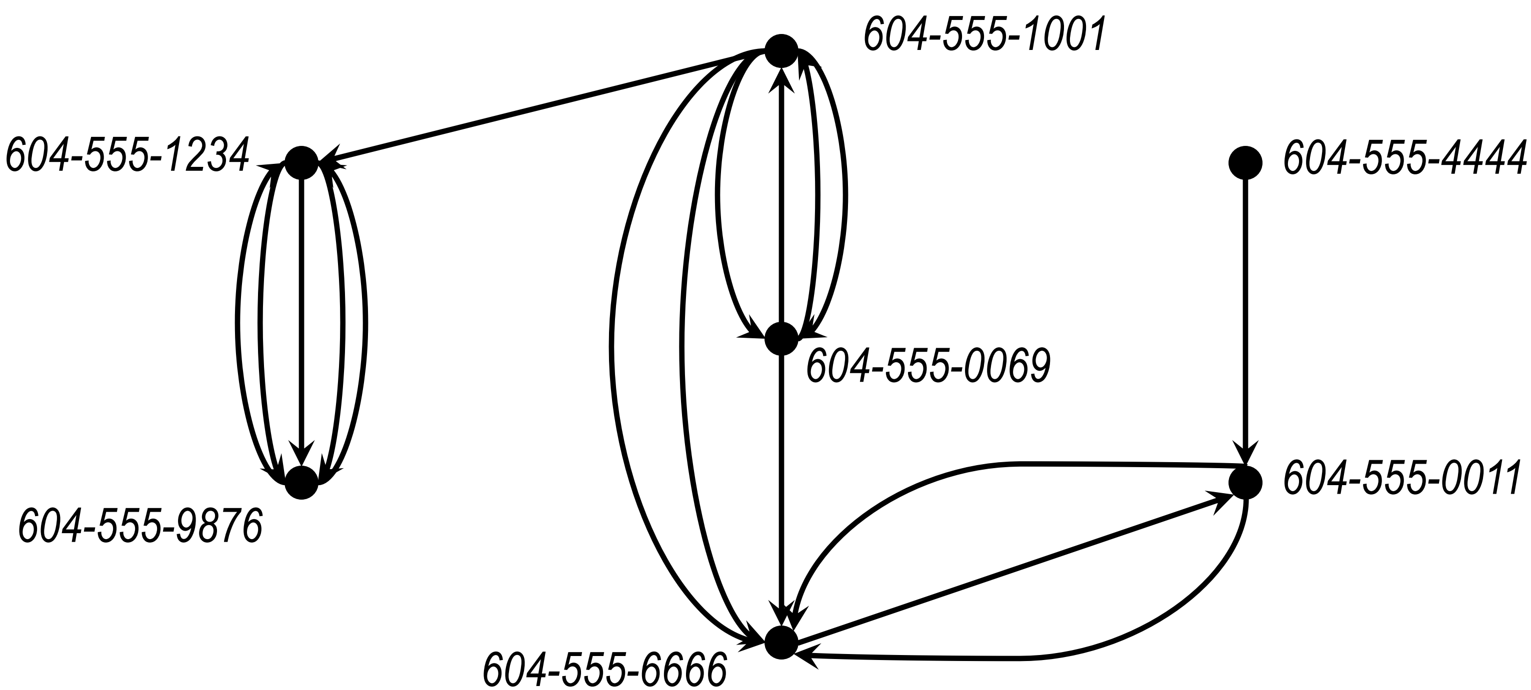
Applications: 6 degrees of Kevin Bacon

- *Can model relationships between people using graphs*
- *A person (a group of people) is represented by a vertex. An edge connects two vertices when the people represented by the vertices know each other.*
- *Conjecture: Every pair of people are connected by a path of length 6*



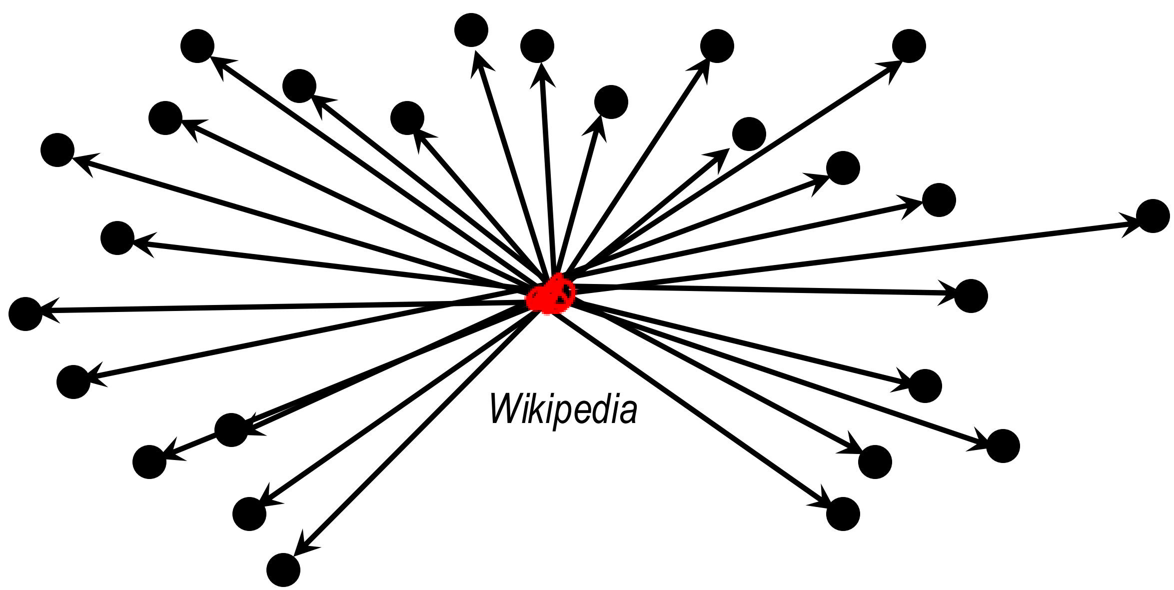
Applications: Call Graphs

- *Graphs can be used to model telephone calls made in a network, such as long distance telephone network.*
- *A directed multigraph can be used to model calls where each telephone number is represented by a vertex and each telephone call is represented by a directed edge.*



Applications: The Web Graph

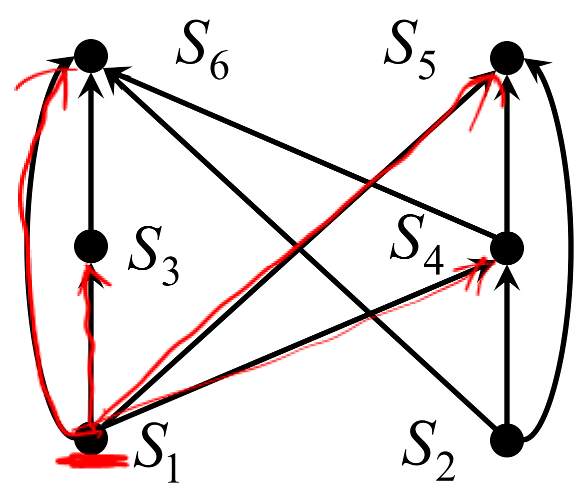
- *The World Wide Web can be modeled as a directed graph where each web page is represented by a vertex and where an arc starts at the web page a and ends at the web page b if there is a link on a pointing to b.*



Applications: Dependence graphs

- *Dependence graphs are used throughout computer science to analyze and optimize computations*
- *For instance, programs can be executed more rapidly by executing certain statements concurrently, but only if they depend only on statements which have already been executed.*
- *A dependence graph represents program statements as nodes, and draws a directed arc whenever statement a must be executed before statement b*

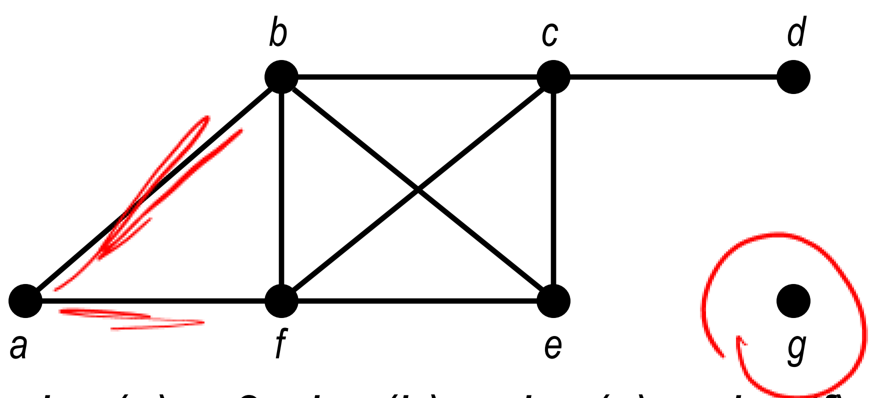
S_1 $a := 0$
S_2 $b := 1$
S_3 $c := a + 1$
S_4 $d := b + a$
S_5 $e := d + 1$
S_6 $e := c + d$

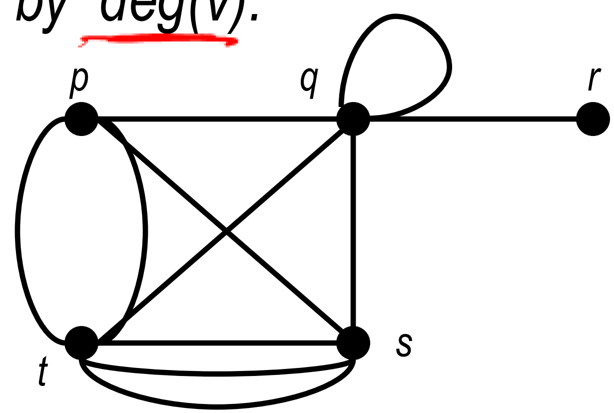


Degree of a Vertex

- *The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex*

  The degree of the vertex v is denoted by deg(v).





- *deg(a) = 2, deg(b) = deg(c) = deg(f), deg(d) = 1, deg(e) = 3, deg(g) = 0*

  = 4
- *deg(p) = 4, deg(q) = deg(t) = 6, deg(r) = 1, deg(s) = 5*
- *A vertex of degree 0 is called* isolated*. g is isolated*
- *A vertex of degree 1 is called* pendant*. d and r are pendant.*

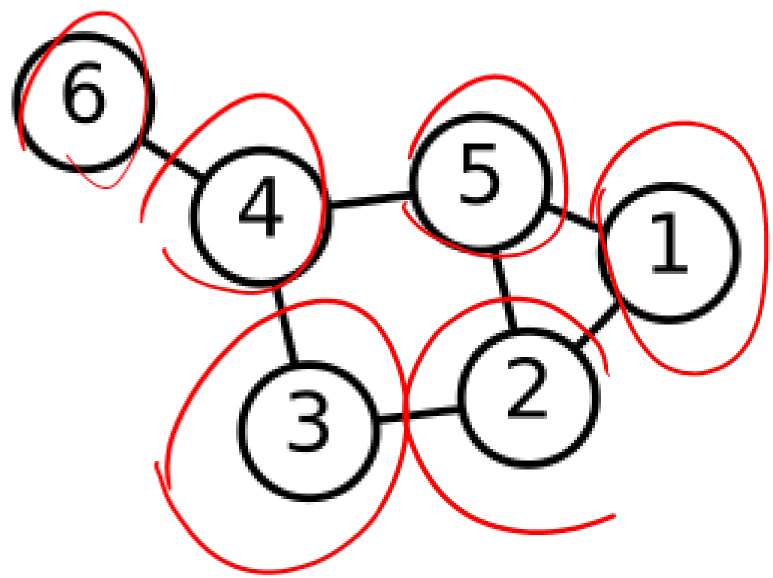
The Handshaking Lemma

- *Let $G = (V, E)$ be an undirected graph with e edges. Then*

$$2e = \sum_{v \in V} deg(v)$$

- *Proof:*

Every edge contributes two to the sum of degrees of the vertices because an edge incident with exactly two (possibly equal) vertices.



The Handshaking Lemma

- *Corollary:*
  An undirected graph has an even number of vertices of odd degree

- *Proof: Let V_1 and V_2 be the sets of vertices of even and of odd degree, respectively, in an undirected graph $G = (V, E)$. Then*

$$2e = \sum_{v \in V} deg(v) = \sum_{v \in V_1} deg(v) + \sum_{v \in V_2} deg(v)$$

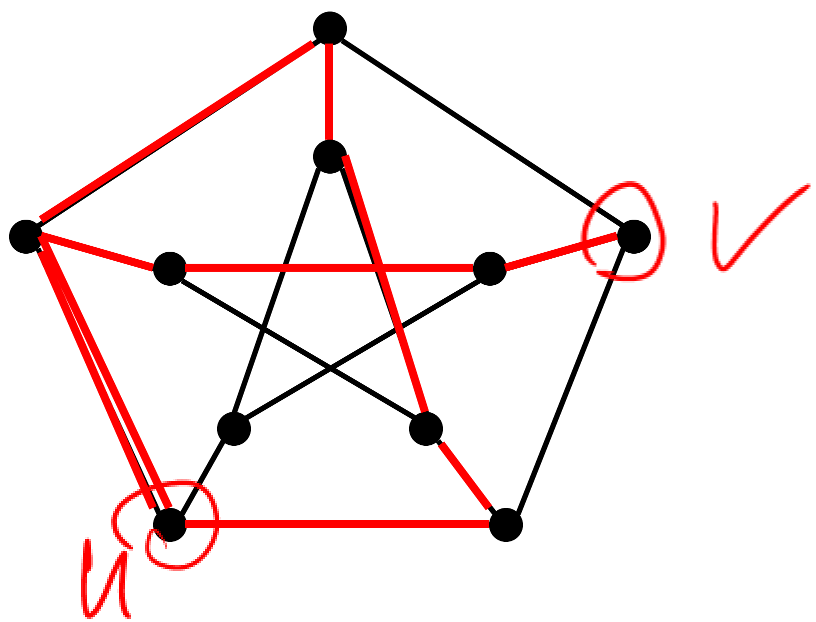
If $|V_2|$ is odd, then the sum is odd, hence a contradiction.

Walks and Paths

- *Let u, v be (not necessarily distinct) vertices in an undirected graph G = (V, E). A u-v walk in G is a finite alternating sequence*

$$u = u_0, e_1, u_1, e_2, u_2, e_3, \ldots, e_{n-1}, u_{n-1}, e_n, u_n = v$$

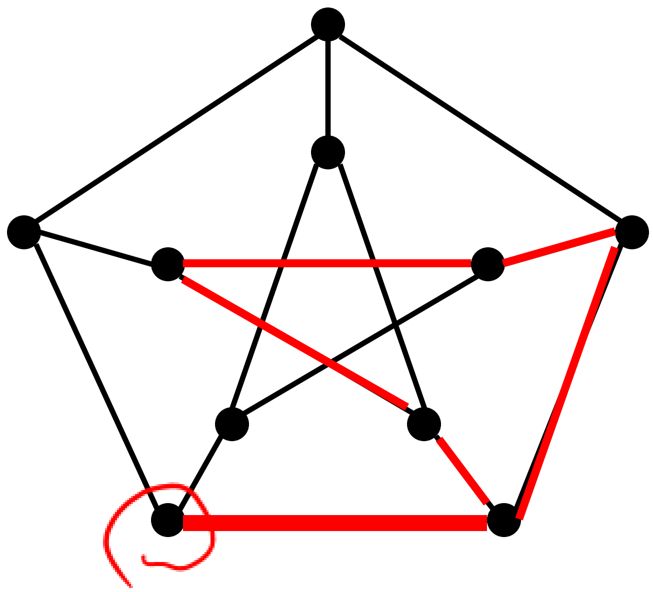
of vertices and edges from G, starting at vertex u and ending at vertex v and involving the n edges $e_i = \{u_{i-1}, u_i\}$, *where* $1 \le i \le n$



Walks and Paths (cntd)

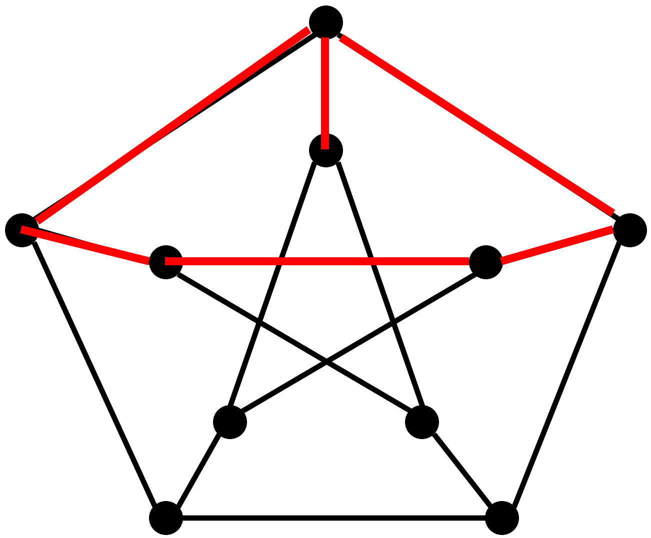
- *The* **length** *of a walk is the number of edges in the walk.*
- *Any u-v walk where u = v (and n > 1) is called a* **closed walk**

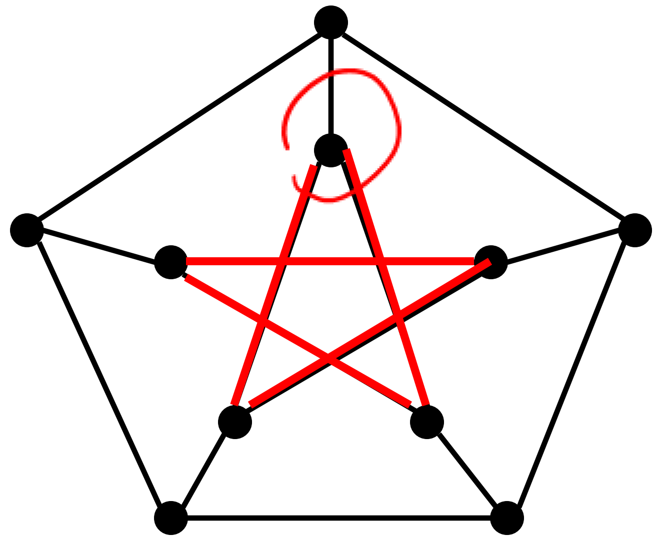
  Otherwise the walk is called **open**.



Walks and Paths (cntd)

- *Consider any u-v walk in an undirected graph G = (V, E)*
- *If no edge in the u-v walk is repeated, then the walk is called a u-v trail. A closed u-u trail is called a circuit*
- *If no vertex of the u-v walk occurs more than once, then the walk is called a u-v path. When u = v, such a closed path is called a cycle*





Walks vs. Paths

- **Theorem.**

  Let $G = (V, E)$ be an undirected graph, with $u, v \in V$, $u \neq v$. If there exists a walk (in G) from u to v, then there is a path (in G) from u to v.

- *Proof:*

  Take a walk from u to v of shortest length

$$u = u_0, e_1, \ldots, u_{k-1}, e_k, u_k, e_{k+1}, u_{k+1}, \ldots, u_{m-1}, e_m, u_m, e_{m+1}, u_{m+1}, \ldots, e_n, u_n = v$$

  If this is a path then we are done.

  Otherwise for some k and m, $k \neq m$, we have $u_k = u_m$

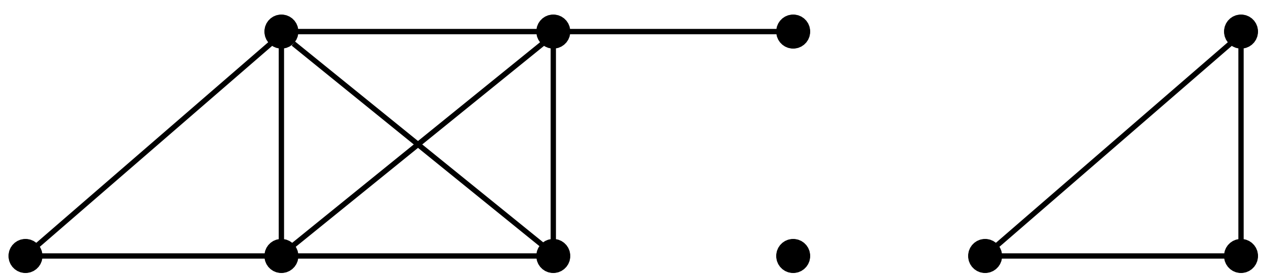
  However, in this case the sequence

$$u = u_0, e_1, \ldots, u_{k-1}, e_k, u_k, e_{m+1}, u_{m+1}, \ldots, e_n, u_n = v$$

  is a walk from u to v of shorter length. A contradiction.

Connected Components

- *An undirected graph $G = (V, E)$ is said to be* **connected** *if there is a path between any two distinct vertices of G*
- *Let $G = (V, E)$ be an undirected graph. A maximal (with respect to inclusion) set of vertices W such that there is a path between any two distinct vertices from W is called a* **connected component** *of G*



Examples

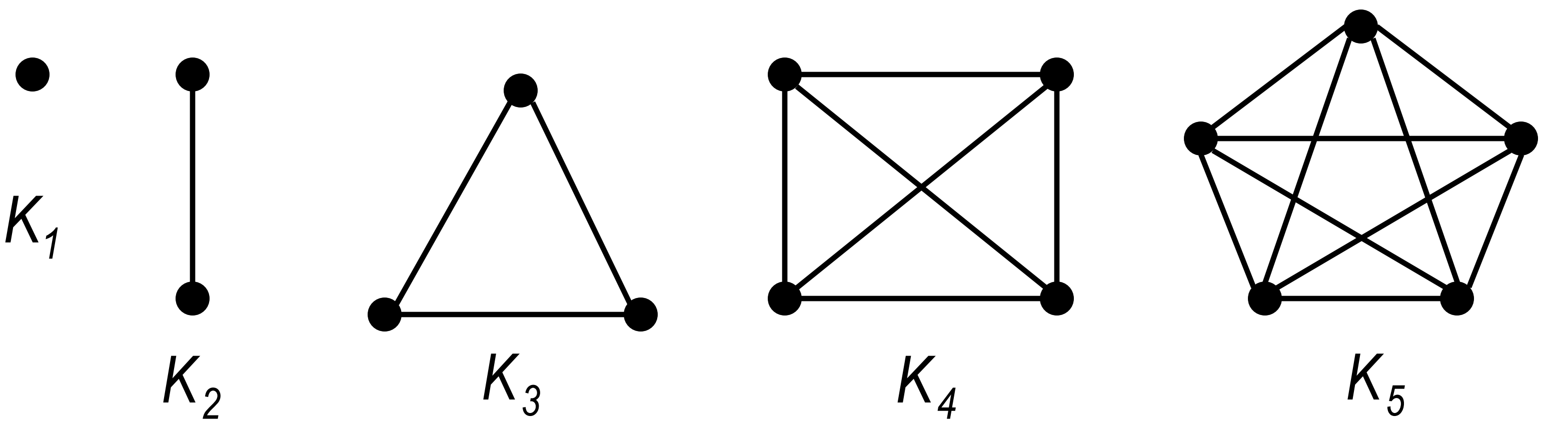
- *A path in an acquaintanceship graph is a chain of people such that every two people adjacent in the chain know each other. It is thought that the shortest path between any two people in the global acquaintanceship graph has length at most 6.*
- *A Collaboration graph is a graph whose vertices represent scientists and an edge connects two vertices if the scientists represented by the vertices coauthor a paper.*
- *The Erdős number of a scientist A is the length of the shortest path between A and the vertex representing mathematician Paul Erdős*
- *My Erdős number is 4*
- *Andrei's is 3*
- *Pavol Hell's is 1*

Examples

- *The Hollywood graph represents actors by vertices and connects two vertices when the actors represented by these vertices have worked together on a movie.*
- *According to IMDB, in Jan. 2006 the Hollywood graph had 637099 vertices representing actors who have appeared in 339896 films, and had more than 20 million edges.*
- *In the Hollywood graph, the Bacon number of an actor c is defined to be the length of the shortest path connecting c and Kevin Bacon.*
- *The Erdős-Bacon number is the sum of one's erdos and bacon numbers*
- *Notable examples: Carl Sagan (7), Colin Firth (6), Kristen Stewart (7), Natalie Portman (7), Sergey Brin (5), Bill Gates (6)*

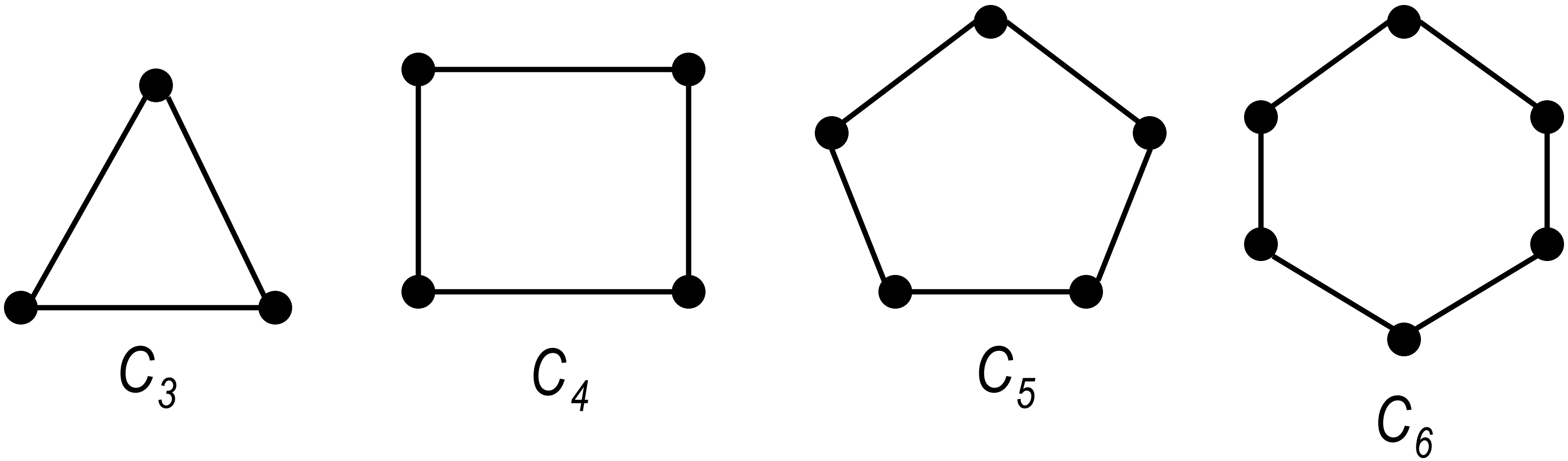
Some Special Graphs

- *Complete Graphs. The complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices*



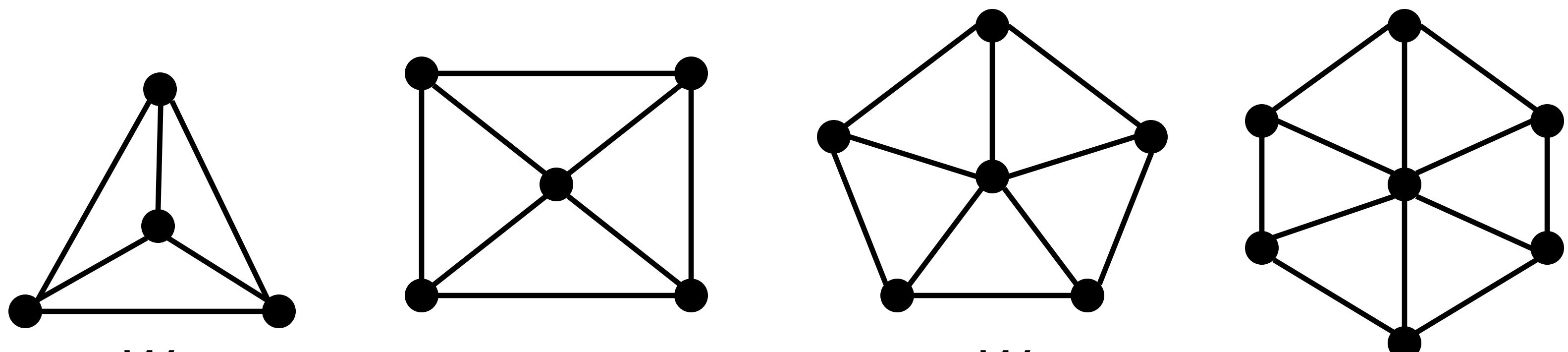
Some Special Graphs (cntd)

- *Cycles*. *The cycle* C_n, $n \geq 3$, *consists of* n *vertices* $v_1, v_2, \ldots, v_n$ *and edges* $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, *and* $\{v_n, v_1\}$.



Some Special Graphs (cntd)

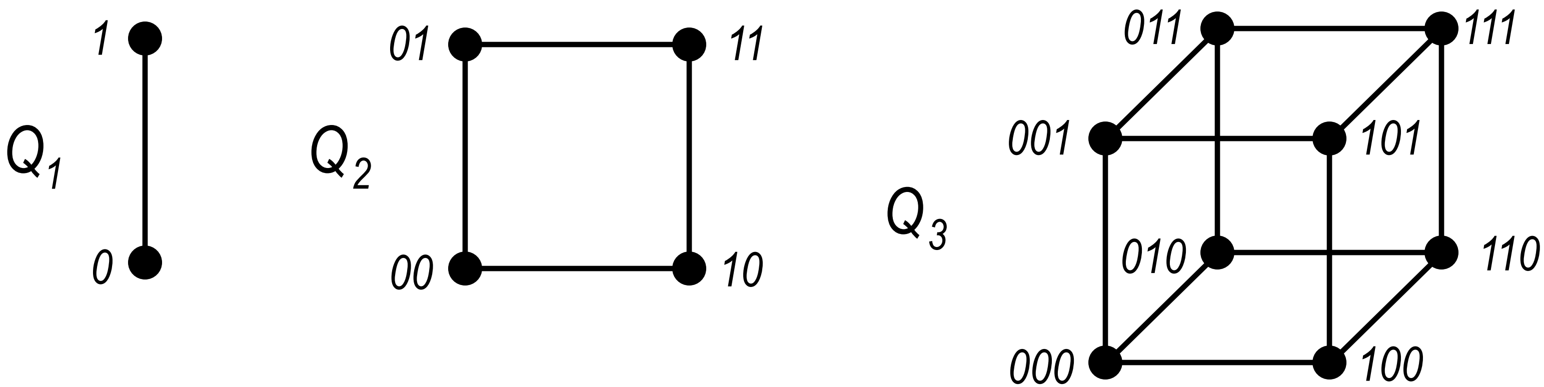
- *Wheels. The wheel, W_n, is obtained from the cycle C_n when we add an additional vertex to it and connect this vertex to each of the n vertices in C_n by new edges*



W_3 W_4 W_5 W_6

Some Special Graphs (cntd)

- *Hypercubes. The n-dimensional hypercube, or n-cube, denoted by Q_n, is the graph that has vertices representing the bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.*



Practice

Exercises from the Book:

7th edition: 1 – 3, 7, 12, 20a-e (page 665), 1, 6, 19 (page 689 – 690)

8th edition: 1 – 3, 7, 12, 20a-e (page 699 – 700), 1, 6, 19 (page 724 – 726)