Graphs II

Discrete Mathematics

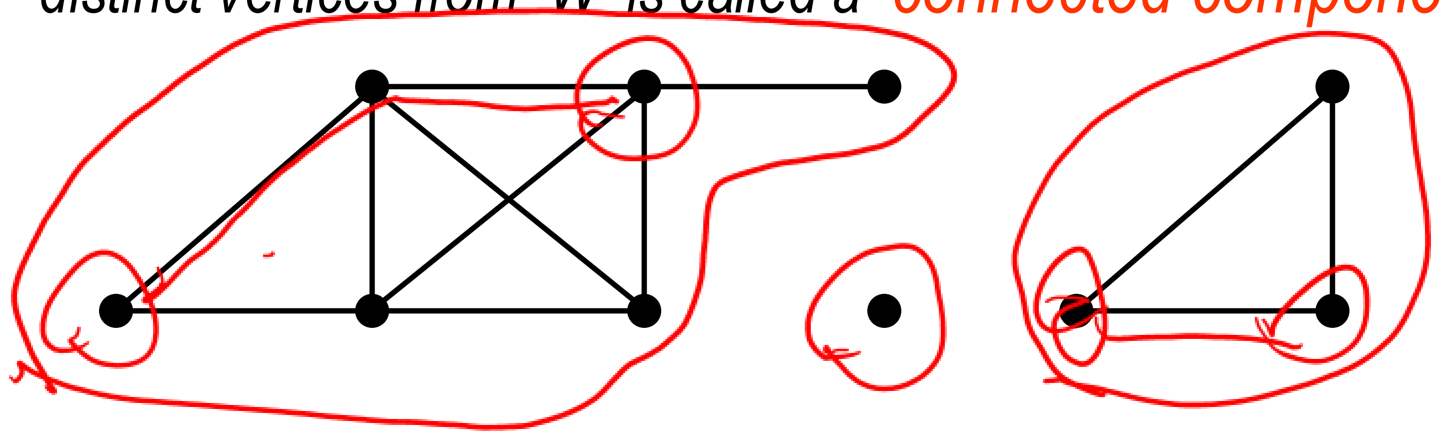
Courtesy of Andrei Bulatov

Previous Lecture

- *Graphs, directed and undirected*
- *Adjacency, incidence, endpoints*
- *Degree of vertices, isolated and pendant vertices*
- *Walks, trails, and paths*
- *Bacon numbers, Erdős numbers:*
  - *Length of shortest path connecting someone to Kevin Bacon in the Hollywood graph (or mathematician Paul Erdős in the Academic co-author graph)*
  - *My Erdos number is 4 (but my Bacon number is ∞...)*
  - *Andrei's Erdos number is 3*
  - *Pavol Hell (professor emeritus of CS here) has Erdos number 1*

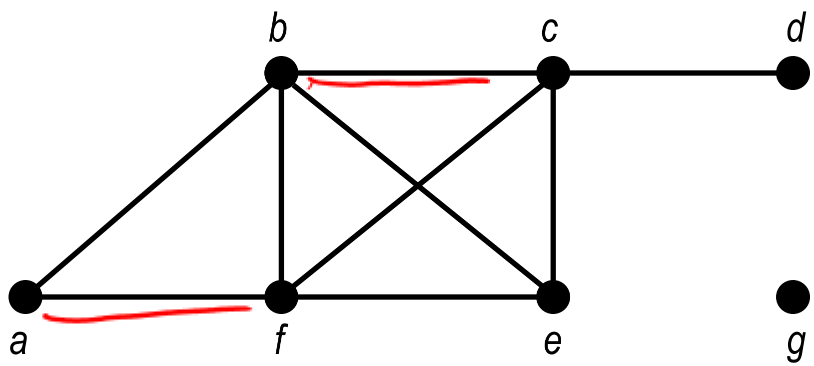
Connected Components

- *Recall:*
  - *A walk is a sequence of alternating vertices and edges*
  - *A path is a walk with no repeated vertices*
- *An undirected graph $G = (V, E)$ is said to be connected if there is a path between any two distinct vertices of G*
- *Let $G = (V, E)$ be an undirected graph. A maximal (with respect to inclusion) set of vertices W such that there is a path between any two distinct vertices from W is called a connected component of G*



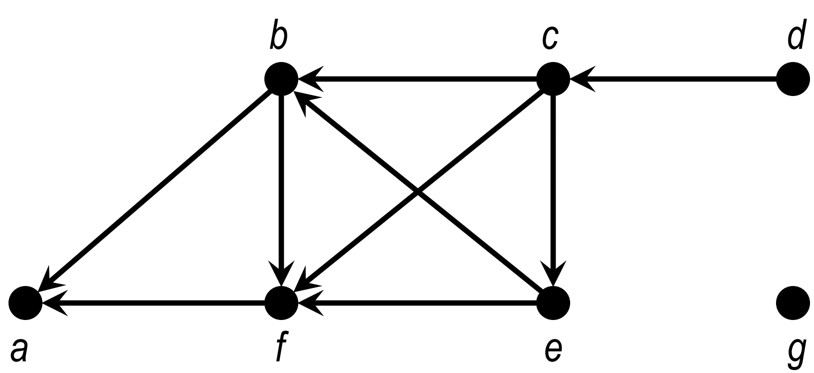
Representation of Graphs: Lists of Vertices and Edges

- *Sets of vertices and edges*



{a, b, c, d, e, f, e, g}

{ {a,b}, {b,c}, {a,f}, {b,f}, {f,c},
{b,e}, {c,e}, {c,d} }, {f,e}}

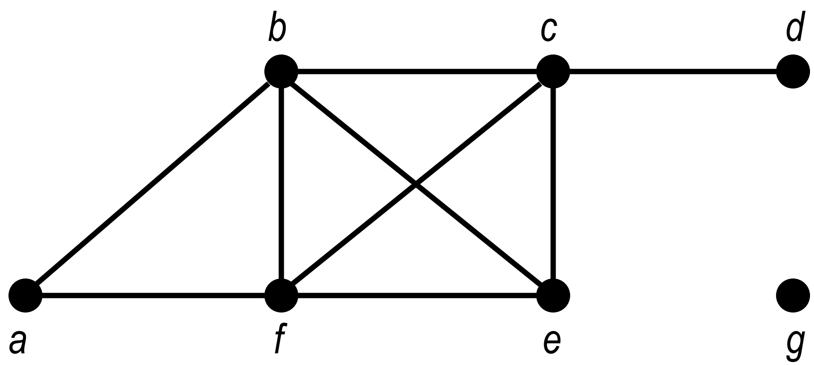


{a, b, c, d, e, f, e, g}

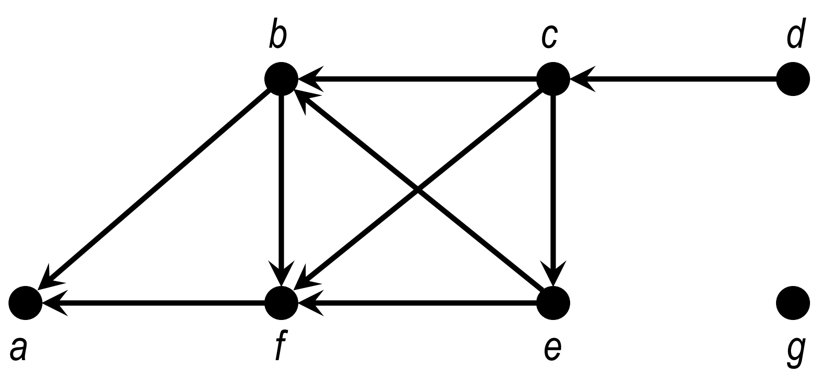
{ (b,a), (c,b), (f,a), (b,f), (c,f),
(e,b), (c,e), (d,c) }, (e,f)}

Representation of Graphs: Adjacency Lists

- *Adjacency list is a table containing for each vertex a set of vertices adjacent to it.*



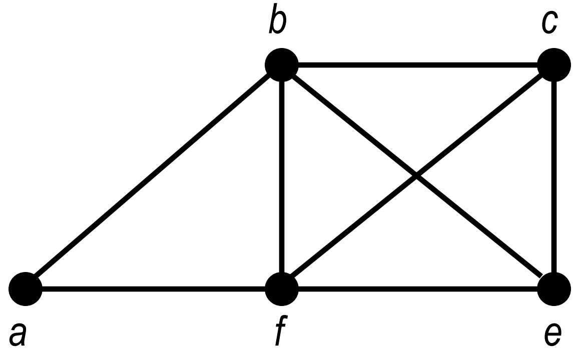
vert.	adjacent	vert.	adjacent
a	b, f	e	b, c, f
b	a, f, c, e	f	a, b, c, e
c	b, d, e, f	g	
d	c		



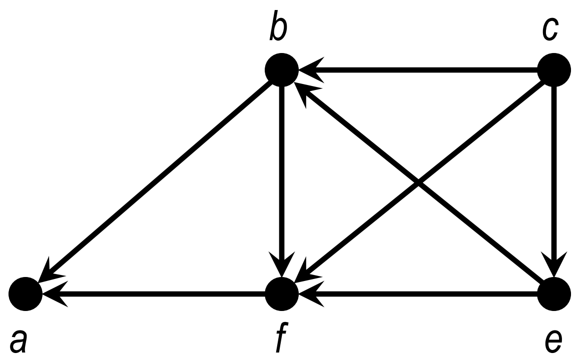
vert.	adjacent	vert.	adjacent
a		e	b, f
b	a, f	f	a
c	b, f, e	g	
d	c		

Representation of Graphs: Adjacency Matrix

- *The adjacency matrix of a graph is a 0-1-matrix whose rows and columns are labeled by the vertices of the graph. The entry in ith row and jth column is 1 if there is an edge from vertex i to vertex j. Otherwise it is 0.*



$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ e \\ f \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

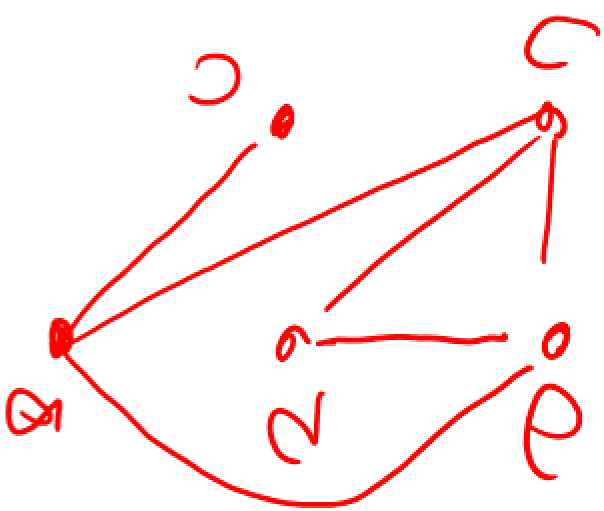


$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ e \\ f \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Example

- *Draw a graph with the adjacency matrices*

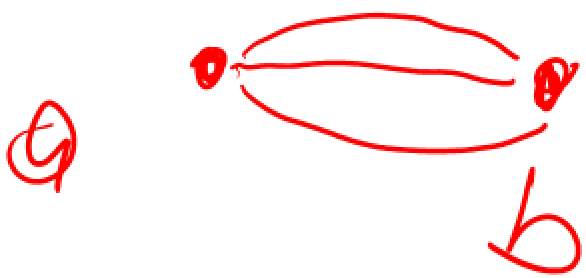
$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array} \qquad g \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$



Example

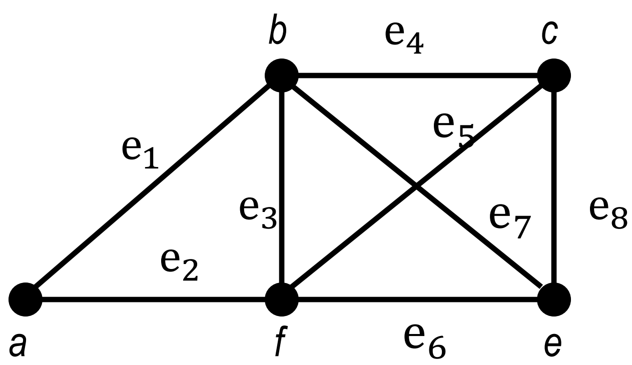
- *The adjacency matrix of a multigraph shows how many edges connect every pair of vertices.*
- *Draw a multigraph with the adjacency matrix*

$$\begin{pmatrix} 0 & 3 & 0 & 2 & 1 \\ 3 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 2 \\ 2 & 1 & 2 & 0 & 3 \\ 2 & 0 & 2 & 3 & 0 \end{pmatrix}$$



Representation of Graphs: Incidence Matrix

- *The incidence matrix of a graph $G = (V, E)$ is a 0-1-matrix whose rows are labeled by vertices of G and columns are labeled by edges of G. The entry in ith row and jth column is 1 if the vertex i is incident to the edge j.*

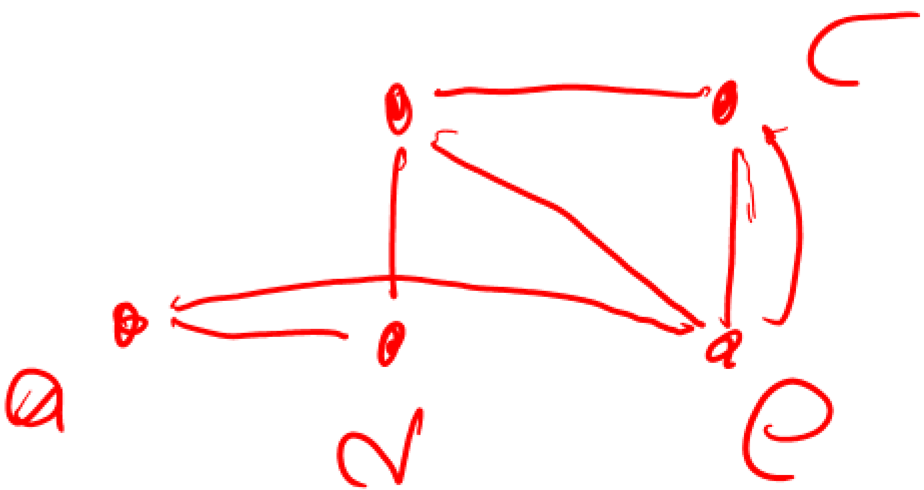


$$\begin{array}{c} \\ a \\ b \\ c \\ e \\ f \end{array}\begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Example

- *Draw a graph with the incidence matrix*

$$\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$



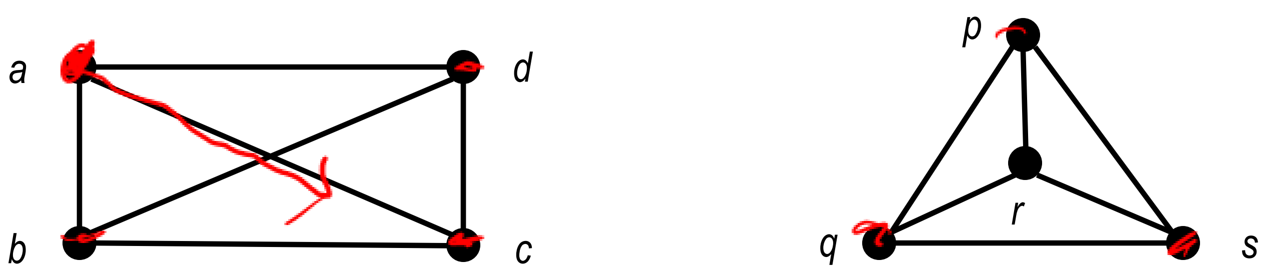
Comparing representations

- *Which method is better?*
- *It depends on a graph and the problem we are going to solve.*
- *If a graph is sparse (e.g. constant number of edges per vertex), then the adjacency list and incidence matrix are more concise.*
- *If a graph is dense, that is, contains many edges, then adjacency matrix is more concise.*
- *Some algorithms may require particular type of representation.*

- *Problem (CONNECTIVITY):*
  Compute the connected components of a graph
- *On an adjacency matrix or incidence matrix, CONNECTIVITY can be solved efficiently by matrix multiplication*

Isomorphism of Graphs

- *We often need to know whether it is possible to draw two graphs in the same way*
- *Simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are* **isomorphic** *if there is a bijection f from V_1 to V_2 such that vertices a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2*
- *Such a function f is called an* **isomorphism**.



x	a	b	c	d
f(x)	r	q	s	p

Graph Invariants

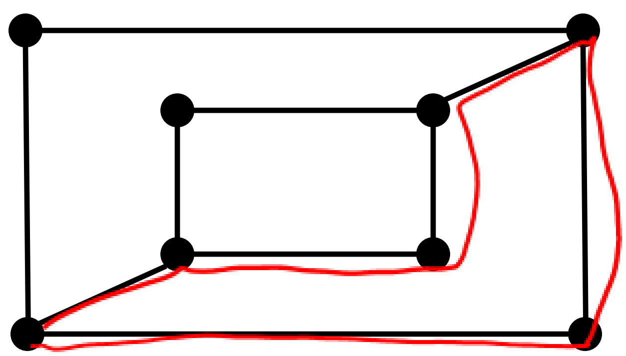
- *In general, it is very hard to decide whether two graphs are isomorphic or not.*
  - *At least as hard as factoring integers, it seems*
- *We can however often prove that two graphs are not isomorphic by finding a property that one graph has which the other does*
- *Example: Are these graphs isomorphic?*

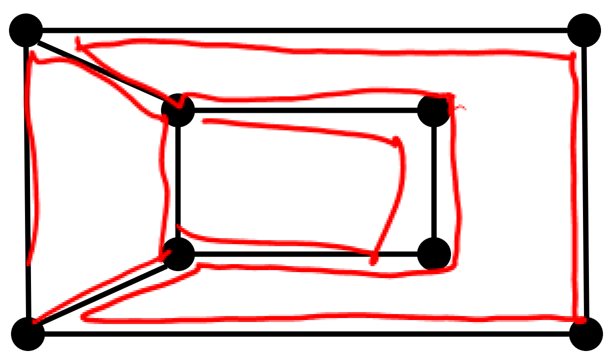
- *A property preserved by isomorphism of graphs is called a graph invariant.*

More invariants

- *Graph invariants include:*
  - *number of vertices & edges*
  - *number of connected components*
  - *degrees of vertices*
  - *the length of paths and cycles*

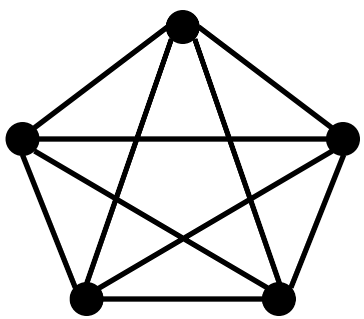
Example: Are these graphs isomorphic?



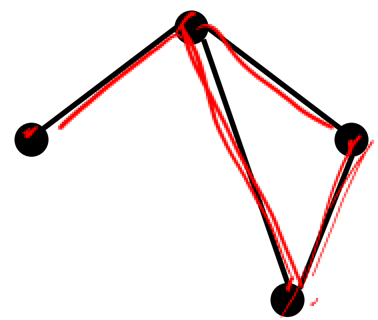


Subgraphs

- *Sometimes we need only a part of a big graph.*
- *A* subgraph *of a graph $G = (V,E)$ is a graph $H = (W,F)$, where $W \subseteq V$ and $F \subseteq E$.*



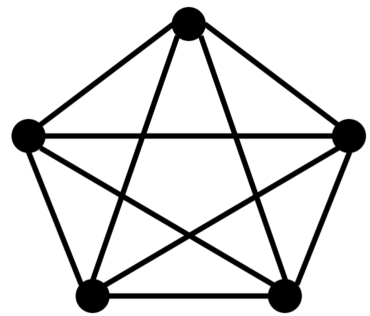
G



H

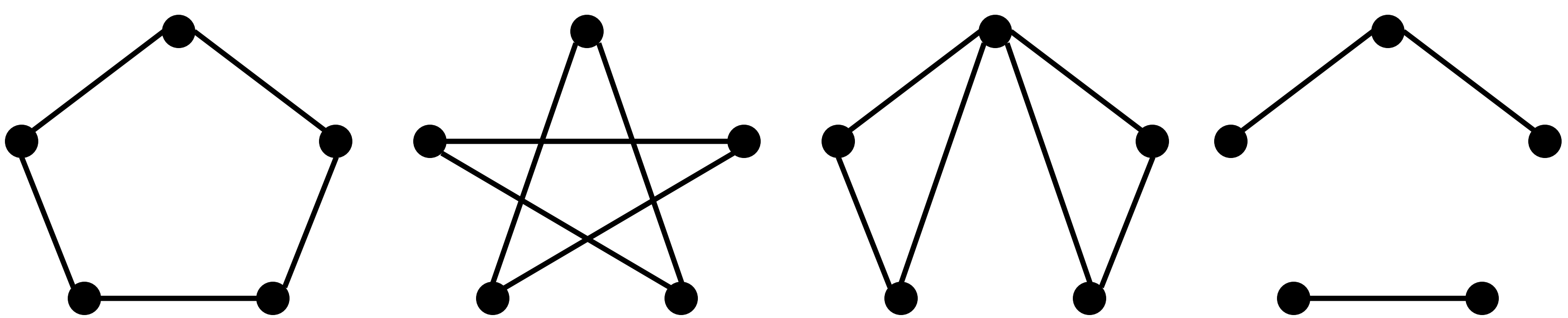
Subgraphs (cntd)

- *If $H = (W,F)$ is a subgraph of $G = (V,E)$ and $W = V$ then H is called a* ***spanning subgraph***



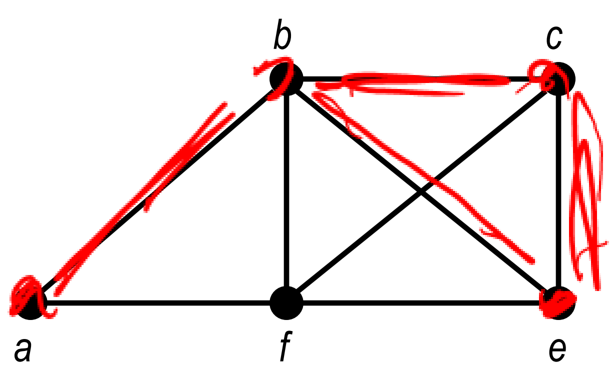
G

Spanning subgraphs:

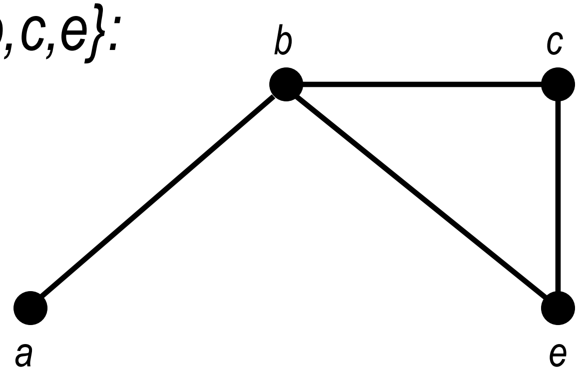


Subgraphs (cntd)

- *Let $G = (V,E)$ be a graph and $W \subseteq V$ a nonempty set of vertices. The* ***subgraph*** *of G* ***induced*** *by W is the subgraph whose vertex set is W and which contains all edges (from G) whose endpoints are in W.*
- *A subgraph H of a graph $G = (V,E)$ is called an induced subgraph if there is $W \subseteq V$ such that H is the subgraph induced by W.*

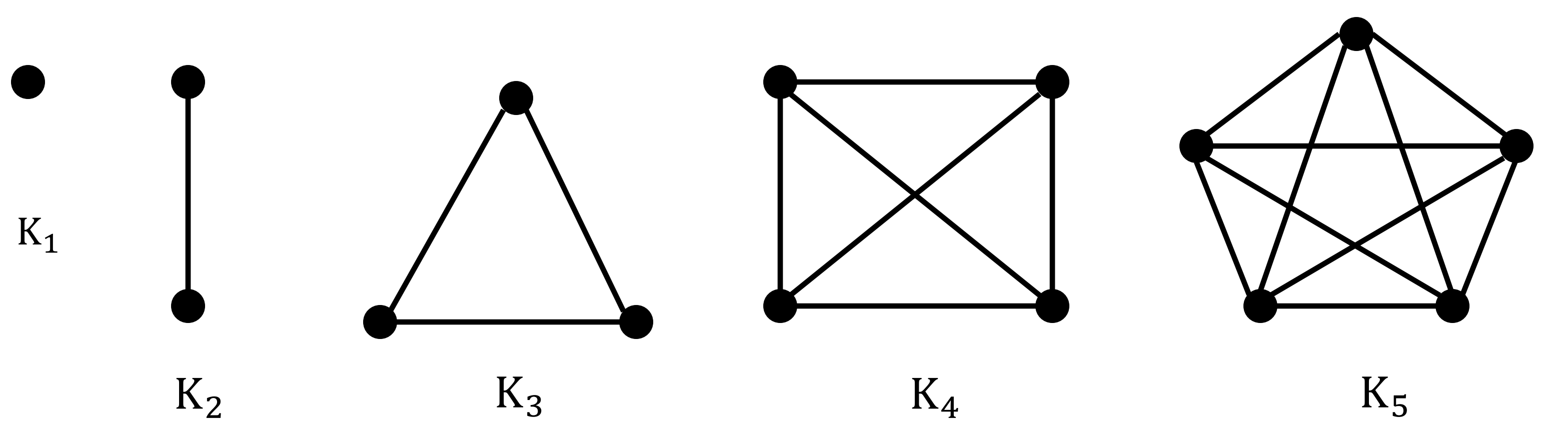


$W = \{a,b,c,e\}$:



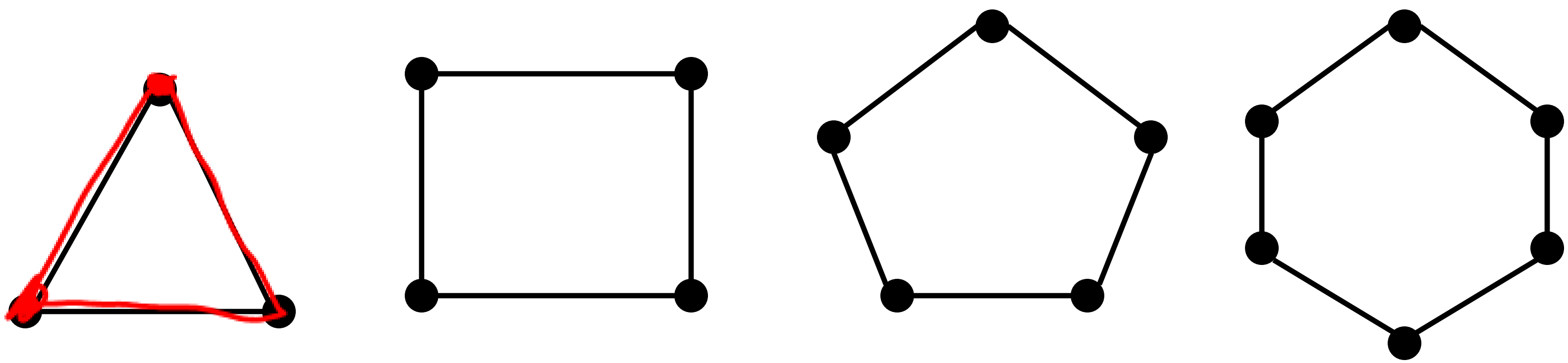
Some Special Graphs

- *We're often interested in whether a graph (or subgraph) is isomorphic to a graph with a particular structure*
- *This tells us a lot about the properties and structure of the graph*
- *Complete Graphs. The complete graph K_n on n vertices is the simple graph that contains exactly one edge between each pair of distinct vertices*



Some Special Graphs (cntd)

- *Cycles. The cycle C_n for $n \geq 3$ consists of n vertices* $v_1, v_2, \ldots, v_n$ *and edges* $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, *and* $\{v_n, v_1\}$.



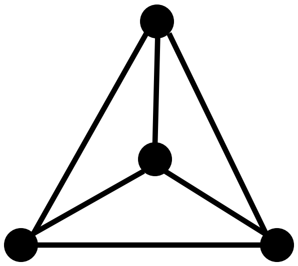
C_3 C_4 C_5 C_6

- *Theorem:*

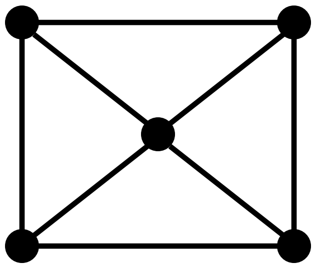
  A graph has a loop of length k if and only if it has a subgraph isomorphic to C_k

Some Special Graphs (cntd)

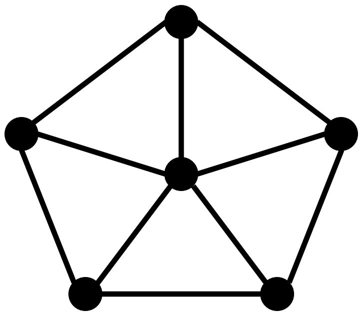
- *Wheels. The wheel,* W_n, *is obtained from the cycle* C_n *when we add an additional vertex to it and connect this vertex to each of the n vertices in* C_n *by new edges*



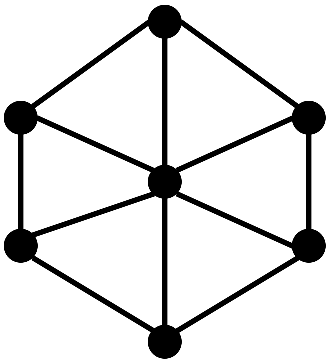
W_3



W_4



W_5

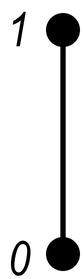


W_6

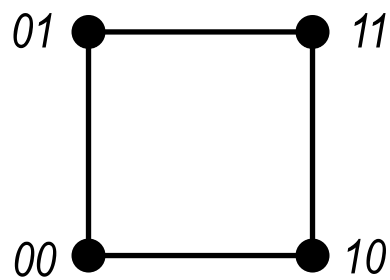
Some Special Graphs (cntd)

- *Hypercubes. The n-dimensional hypercube, or n-cube, denoted by* Q_n, *is the graph that has vertices representing the bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.*

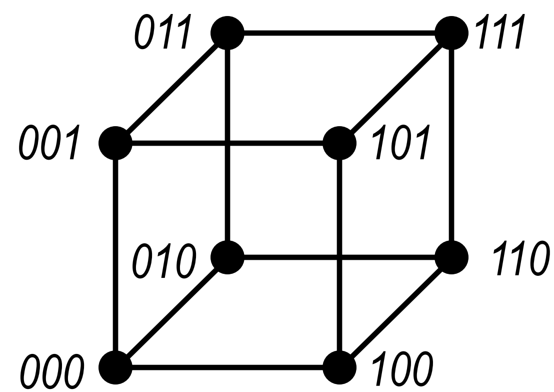
Q_1



Q_2

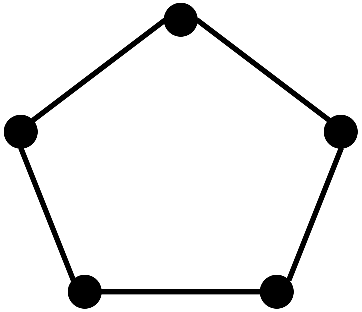


Q_3

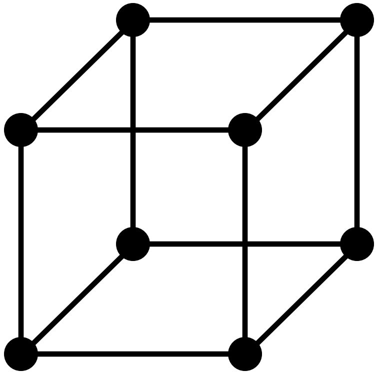


Regular Graphs

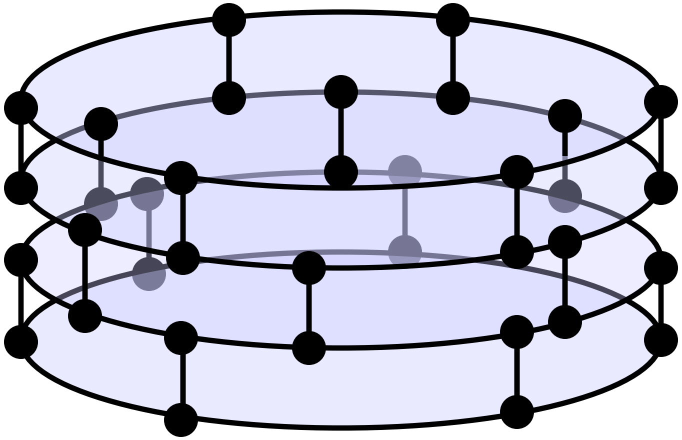
- *A graph is called* *k-regular* *if all its vertices have degree k.*



2-regular graphs are cycles



The k-cube is a k-regular graph



A brick graph, another example of a 3-regular graph

Practice

Exercises from the Book:

7th edition: 26, 53, 55 (page 666 – 667), 1, 3, 5, 7, 9c, 11, 39 (page 675 – 676)

8th edition: 26, 55, 57, (page 699 – 700), 1, 3, 5, 7, 9c, 11, 43 (page 710 – 712)