Bipartite Graphs

Discrete Mathematics

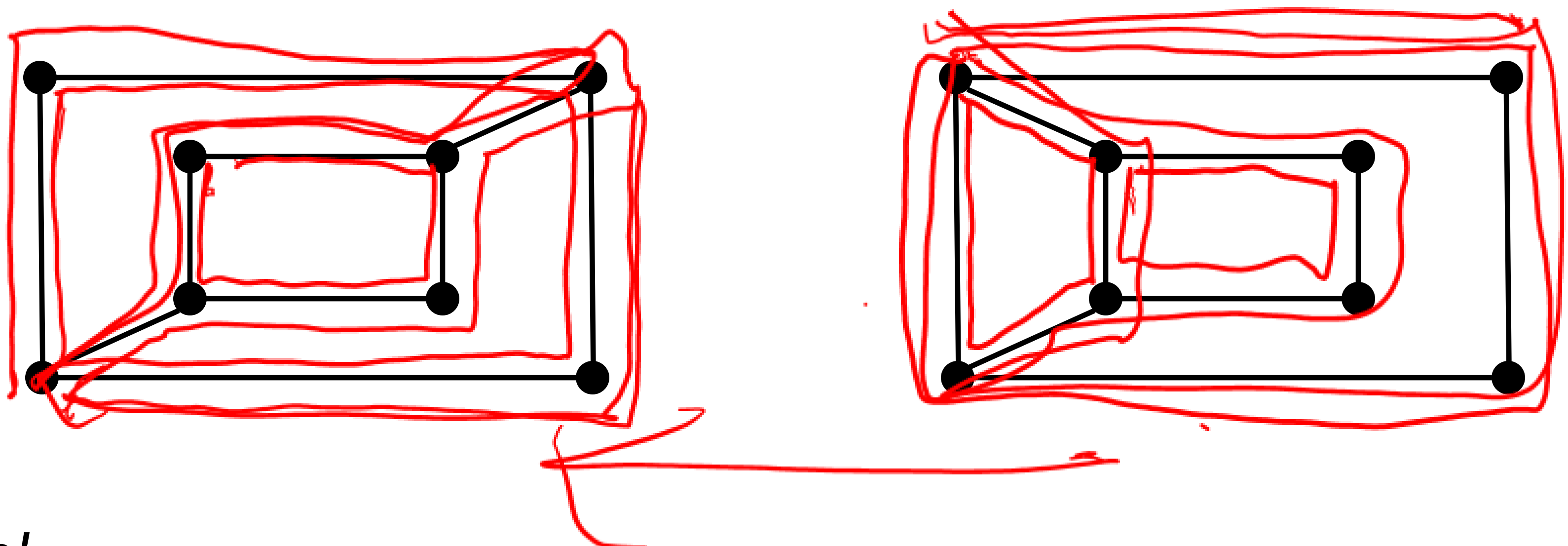
Courtesy of Andrei Bulatov

Previous Lecture

- *Graph representations*
  - *Vertex/edge lists, adjacency lists, adjacency matrices, incidence matrices*
- *Graph isomorphism*
- *Graph invariants*
- *Subgraphs*

Error from last class

Example: Are these graphs isomorphic?



No!

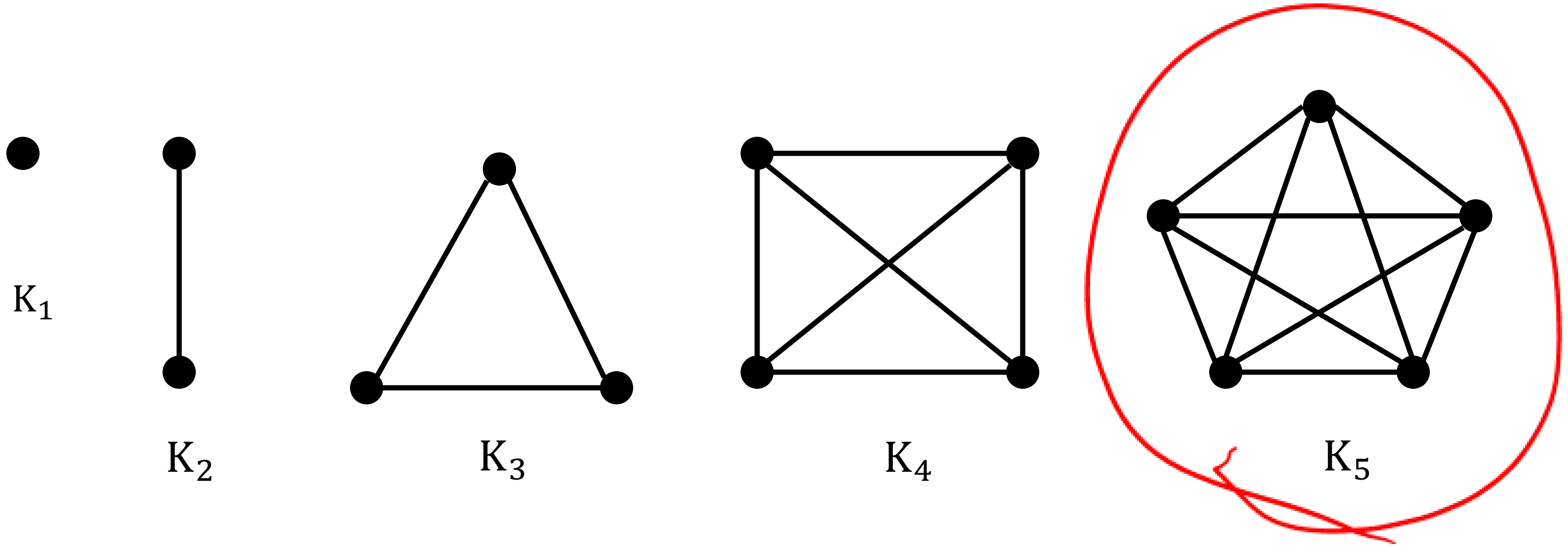
Right one has 3 cycles of length 4, but the left only has 2 of length 4

Left has 4 cycles of length 6, but the right only has 2

Right has a cycle of length 8, left does not

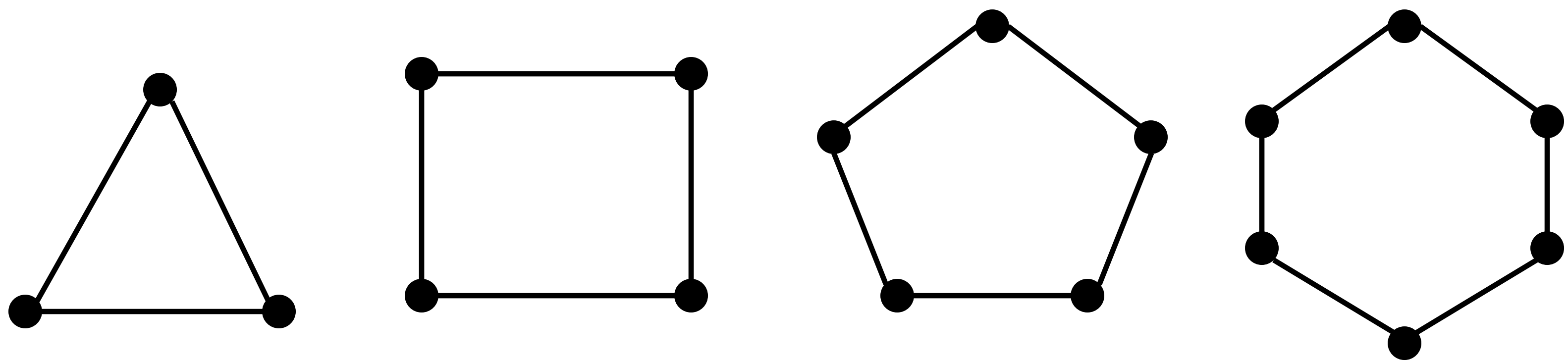
Some Special Graphs

- *Complete Graphs. The complete graph K_n on n vertices is the simple graph that contains exactly one edge between each pair of distinct vertices*



Some Special Graphs (cntd)

- *Cycles.* *The cycle C_n for $n \geq 3$ consists of n vertices* $v_1, v_2, \ldots, v_n$ *and edges* $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, *and* $\{v_n, v_1\}$.



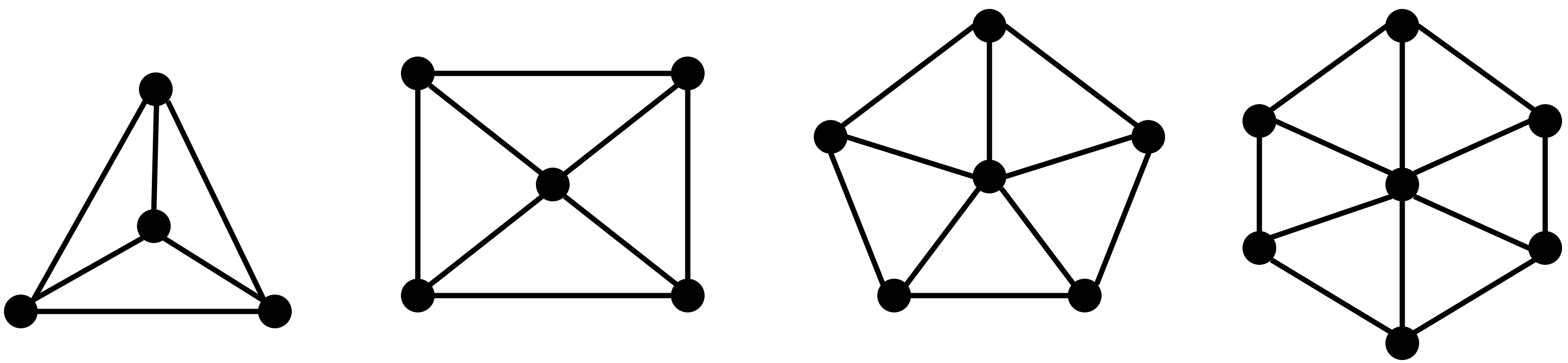
C_3 C_4 C_5 C_6

- *Theorem:*

  A graph has a loop of length k if and only if it has a subgraph isomorphic to C_k

Some Special Graphs (cntd)

- *Wheels. The wheel,* W_n, *is obtained from the cycle* C_n *when we add an additional vertex to it and connect this vertex to each of the n vertices in* C_n *by new edges*



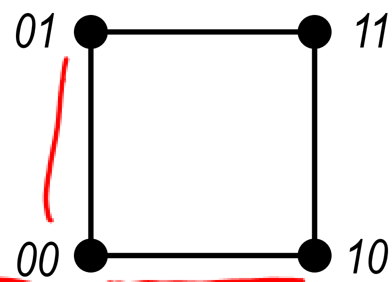
W_3 W_4 W_5 W_6

Some Special Graphs (cntd)

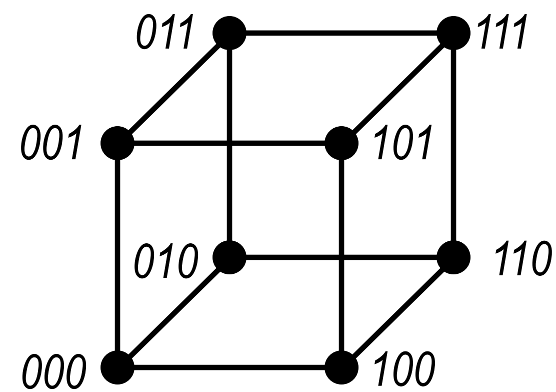
- *Hypercubes*. *The n-dimensional hypercube, or n-cube, denoted by* Q_n, *is the graph that has vertices representing the bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.*

Q_1

Q_2

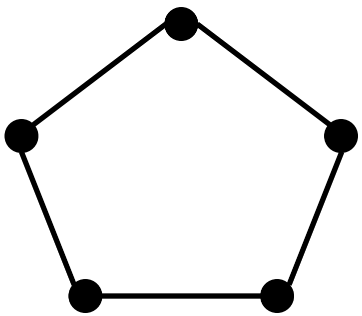


Q_3



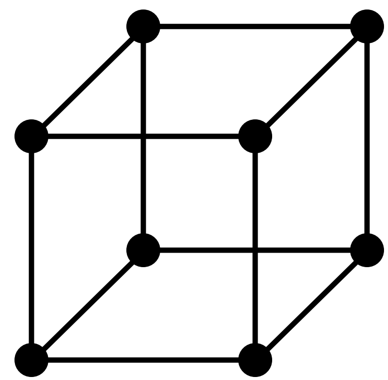
Regular Graphs

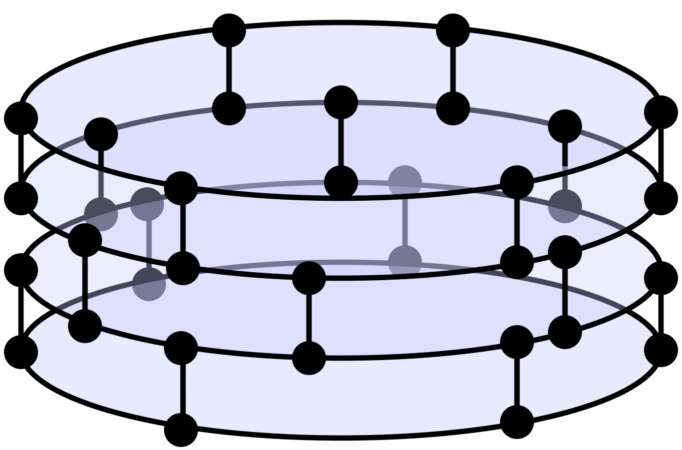
- *A graph is called* ***k-regular*** *if all its vertices have degree k.*



2-regular graphs are cycles

The k-cube is a k-regular graph





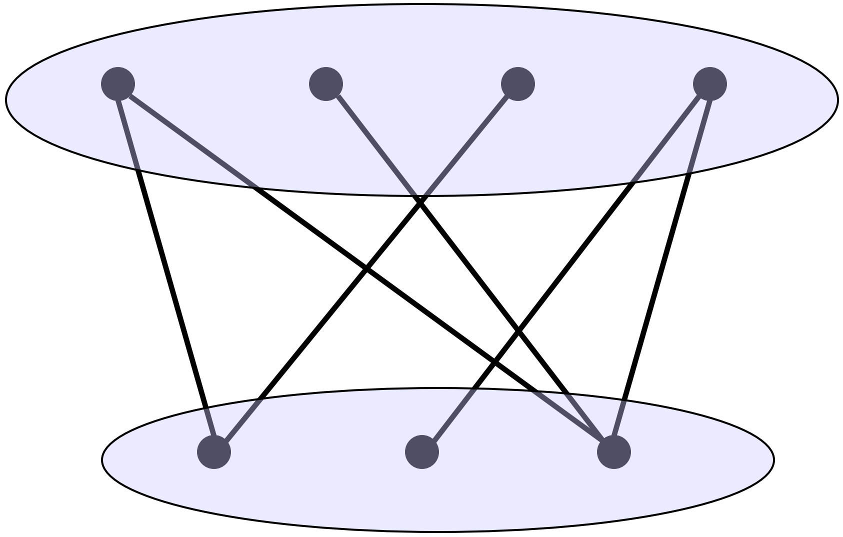
A brick graph, another example of a 3-regular graph

Are they isomorphic?

Bipartite Graphs

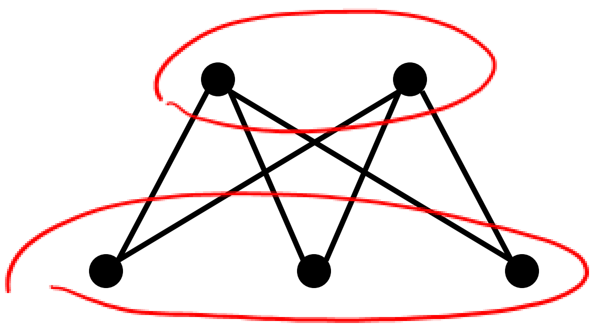
- *A simple graph G is* **bipartite** *if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2.*

  When this condition holds, we call the pair V_1, V_2 a **bipartition** *of the vertex set V of G*

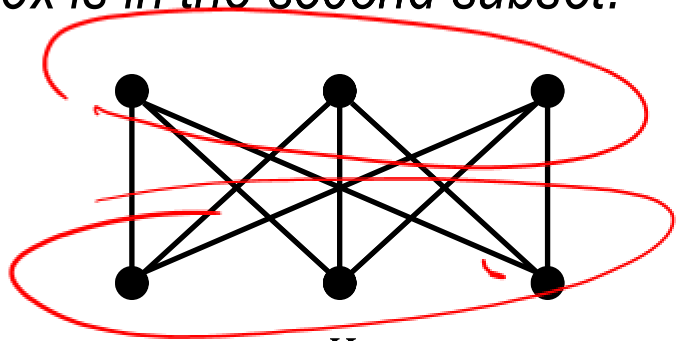


Complete Bipartite Graphs

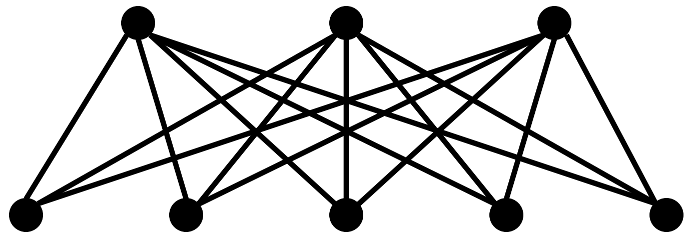
- *The complete bipartite graph $K_{m,n}$ is the graph that has its vertex set partitioned into two subsets of m and n vertices, respectively. There is an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.*



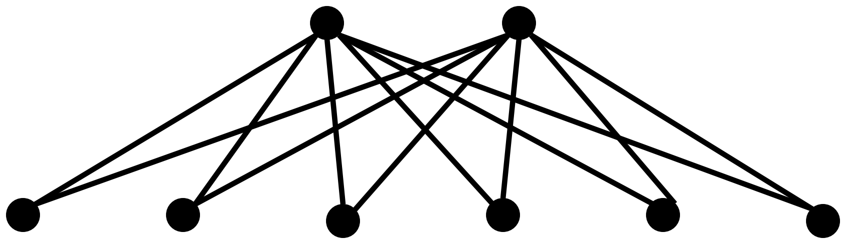
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$



$K_{2,6}$

More Bipartite Graphs

- **Theorem.**

  For a simple graph $G = (V,E)$ the following conditions are equivalent:

  (1) G is bipartite,

  (2) G does not contain a cycle of odd length,

  (3) it is possible to color each vertex of G in one of two different colors so that no two adjacent vertices are colored the same color.

  2-colouring

- *Proof:*

  We prove implications (1) $\to$ (2), (2) $\to$ (3), and (3) $\to$ (1).

More Bipartite Graphs (cntd)

- *(1) → (2) (bipartite implies no odd cycles)*

  Suppose that G is bipartite, and U, W is a bipartition.

  Let $u = u_0, e_1, u_1, \ldots, e_{n-1}, u_{n-1}, e_n, u_n = u$ *be a cycle. Without loss of generality we may assume that* $u \in U$.

  Then $u_1 \in W, u_2 \in U, u_3 \in W, \ldots$ *and in general,* u_i *is in U if and only if i is even.*

  But $u_n = u$ *is in U, so n is even as required.*

More Bipartite Graphs (cntd)

- *(2) $\rightarrow$ (3) (no odd cycles implies 2-colourable)*

  Suppose that every cycle in G has even length.

  We show that the following algorithm constructs a proper coloring of G in two colors

take *any vertex and* **color** *it blue*

repeat *the following* **while** *there is an uncolored vertex*

  take *any vertex v that is already colored*

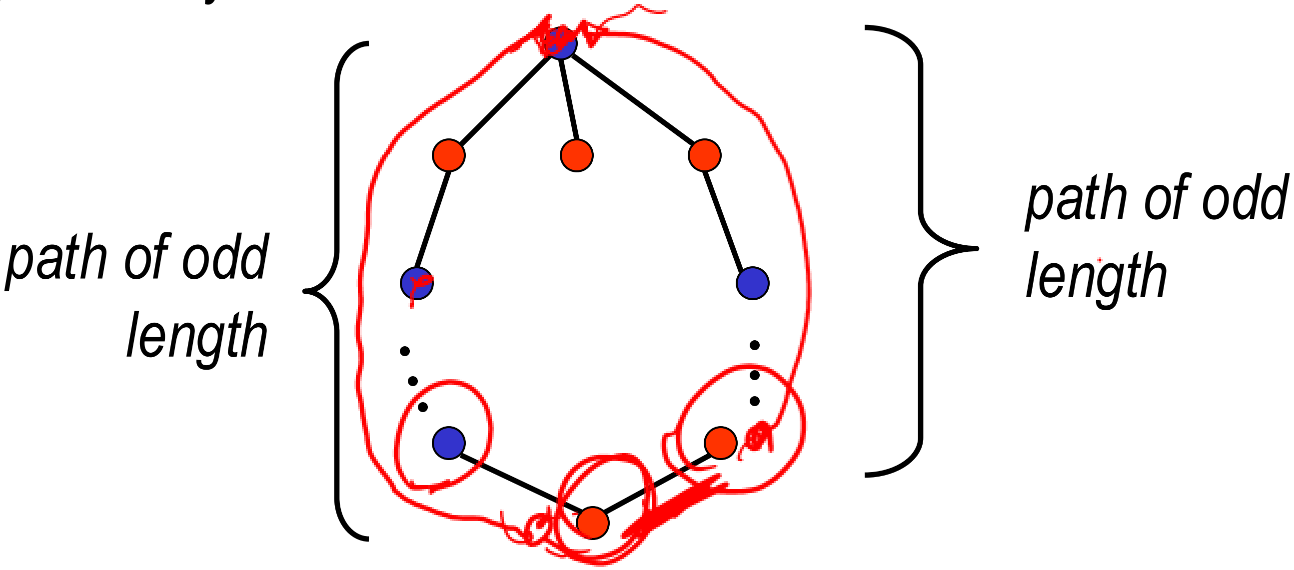
  if *v is colored blue* **then color** *all its neighbors red*

  if *v is colored red* **then color** *all its neighbors blue*

end repeat

More Bipartite Graphs (cntd)

- *To prove the correctness of this algorithm, we have to show that it never attempts to color red a vertex already colored blue, and that it never attempts to color blue a vertex already colored red*
- *Suppose that the algorithm attempts to color blue a vertex previously colored red*



- *Thus, we obtain a cycle of odd length; a contradiction.*

More Bipartite Graphs (cntd)

- *(3) → (1) (2-colourable implies bipartite)*

  Suppose that the vertices of the graph G can be colored in two colors, red and blue, such that no two adjacent vertices receives the same color.

  We construct a bipartition by including all blue vertices into one part of the bipartition, and all red vertices into the other part of the bipartition.

Q.E.D.

Practice

Exercises from the Book:

7th edition: 26, 53, 55 (page 666 – 667), 1, 3, 5, 7, 9c, 11, 39 (page 675 – 676)

8th edition: 26, 55, 57, (page 699 – 700), 1, 3, 5, 7, 9c, 11, 43 (page 710 – 712)