Planar Graphs

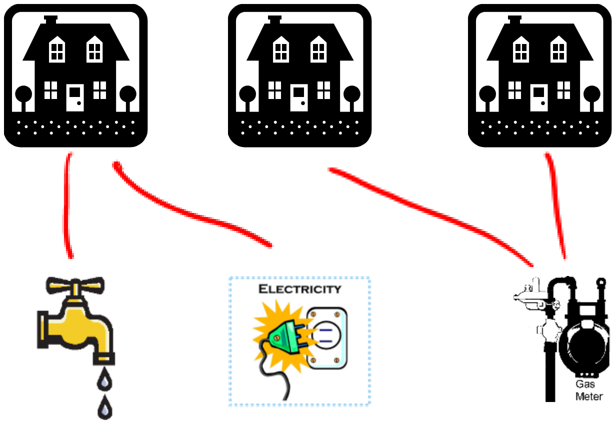
Discrete Mathematics

Courtesy of Andrei Bulatov

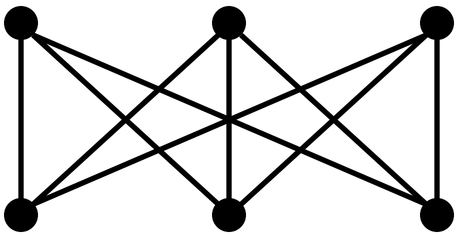
Planar Graphs

- *The Three Utilities and Three Houses puzzle:*

  Is it possible to join three houses to each of three separate utilities so that none of the connections cross?



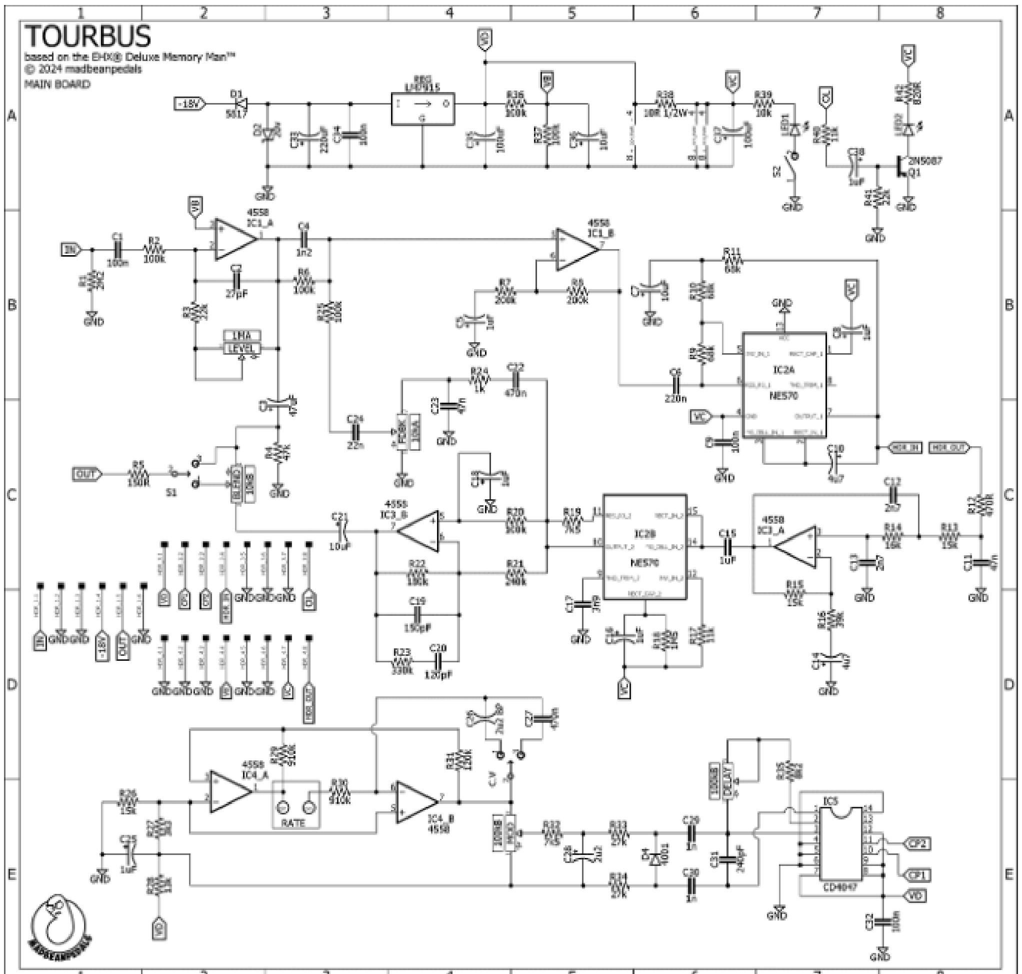
- *This question can be rephrased as: Can the complete bipartite graph $K_{3,3}$ be drawn on the plane such that no two of its edges cross?*

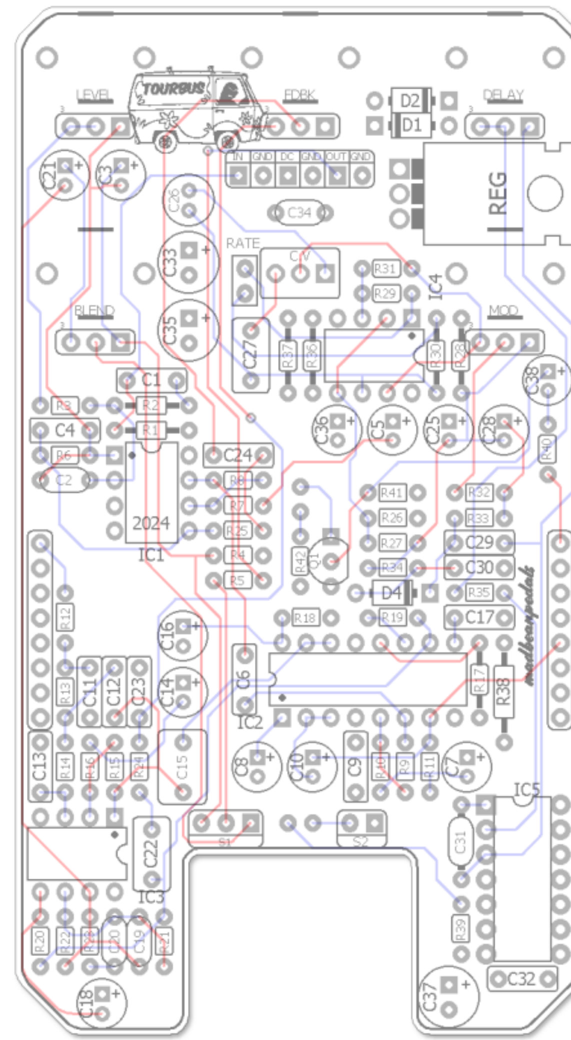


$K_{3,3}$

Digital electronics

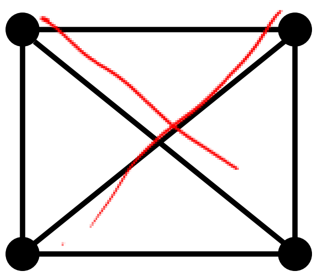
- *How many layers on a PCB are needed for this circuit?*



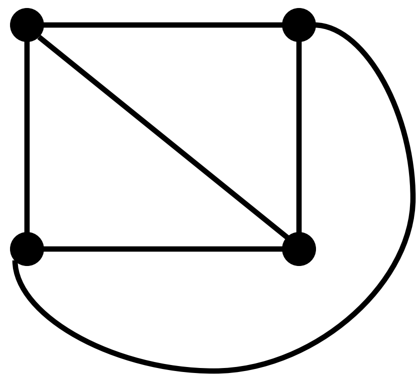


Planar Graphs (cntd)

- *A graph is called* planar *if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of lines or arcs representing them at a point other than their common endpoint). Such a drawing is called a* planar representation *of the graph*



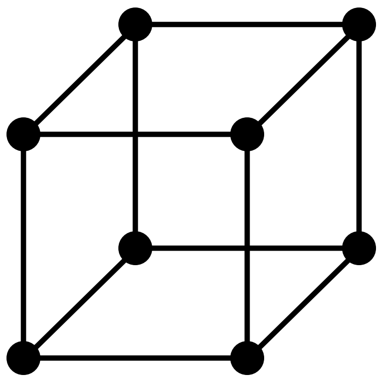
K_4

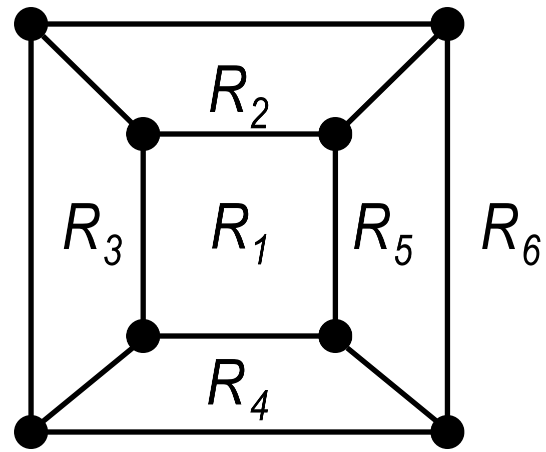


planar representation of K_4

Euler's Formula

- *A planar representation of a graph splits the plane into regions, including the unbounded region.*



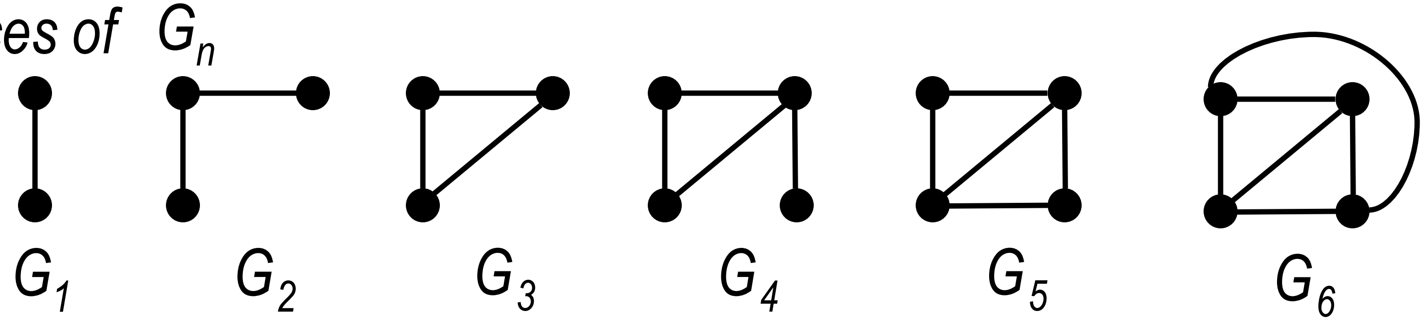


- **Euler's Formula**.

  Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $v - e + r = 2$.

Euler's Formula (cntd)

- *We prove the formula by structural induction.*
- *Fix a planar representation of G.*
- *We construct a sequence of graphs $G_1, G_2, \ldots, G_e = G$ as follows:*
  - *Arbitrarily pick an edge of G to obtain G_1*
  - *Obtain G_n from G_{n-1} by arbitrary adding an edge of G that is incident with a vertex already in G_{n-1}. This construction is possible because G is connected.*
- *G is obtained after e edges are added*
- *Let r_n, e_n, and v_n represent the number of regions, edges, and vertices of G_n*



Euler's Formula (cntd)

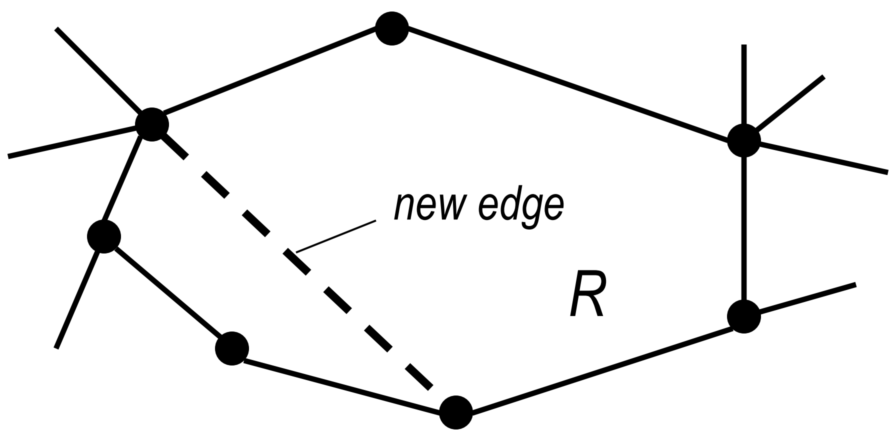
- *Basis step.*

  For G_1 *we have* $r_1 = e_1 = 1$ *and* $v_1 = 2$ *and so* $v_1 - e_1 + r_1 = 2$
- *Inductive step.*

  Suppose that $v_k - e_k + r_k = 2.$ *We will prove* $v_{k+1} - e_{k+1} + r_{k+1} = 2$
- *We consider two cases.*

Euler's Formula (cntd)

- *Case 1. The new edge connect two vertices that are present in* G_k *These two vertices are on the border of the same region because we use a planar representation of G.*

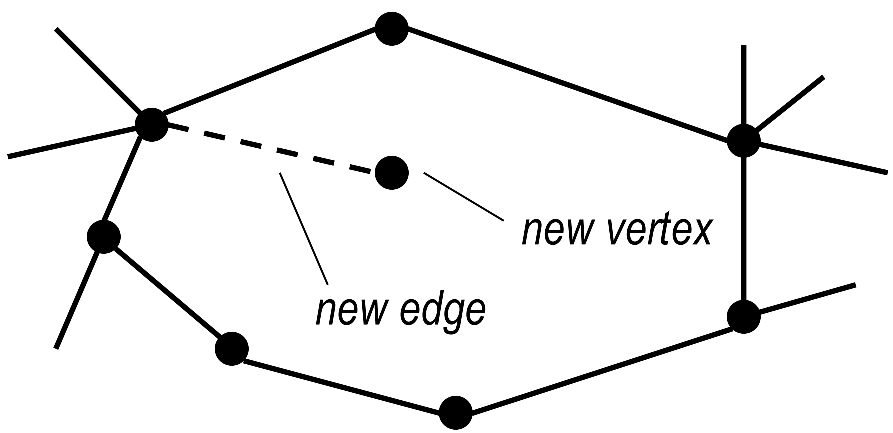


- *The new edge splits this region into two. Therefore* $r_{k+1} = r_k + 1$ $e_{k+1} = e_k + 1$ *and* $v_{k+1} = v_k$. *Thus,*

$$v_{k+1} - e_{k+1} + r_{k+1} = v_k - (e_k + 1) + (r_k + 1) = v_k - e_k + r_k = 2$$

Euler's Formula (cntd)

- *Case 2. The new edge is pendant, and therefore a new vertex is added.*



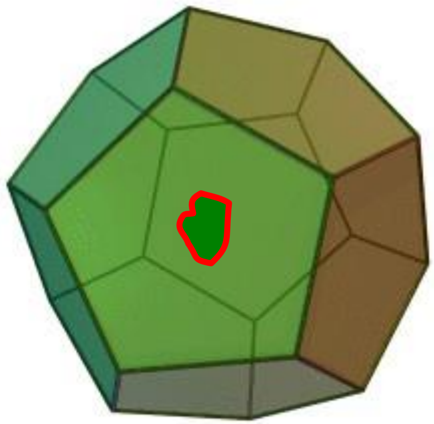
No new region is produced. Therefore $r_{k+1} = r_k$, $e_{k+1} = e_k + 1$, *and* $v_{k+1} = v_k + 1$. *Thus,*

$$v_{k+1} - e_{k+1} + r_{k+1} = (v_k + 1) - (e_k + 1) + r_k = v_k - e_k + r_k = 2$$

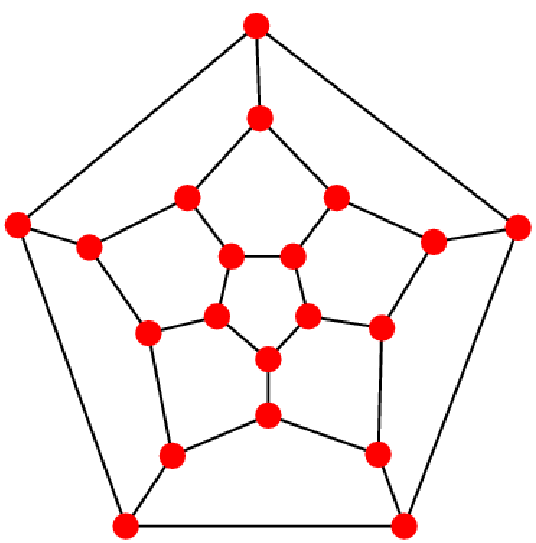
Q.E.D.

Polyhedrons and Planar Graphs

- *Every polyhedron can be converted into a planar graph*



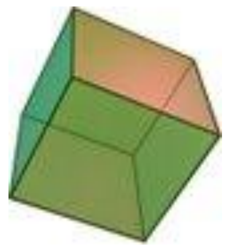
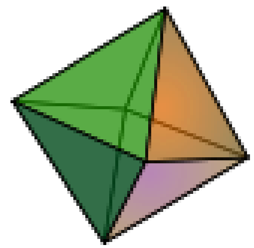
- *Make a hole in the rubber polyhedron*
  Expand it flat
  Put it in the plane
  We obtain a planar graph



- *Vertices become vertices*
  Edges become edges
  Faces become regions

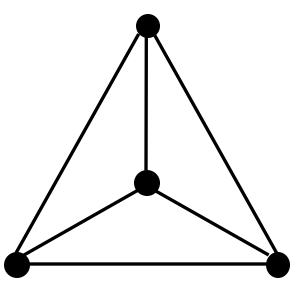
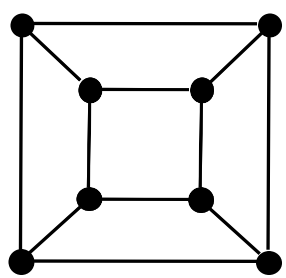
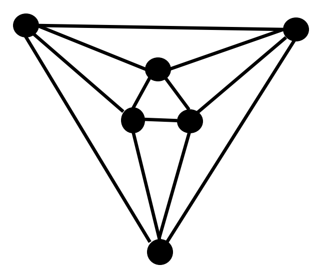
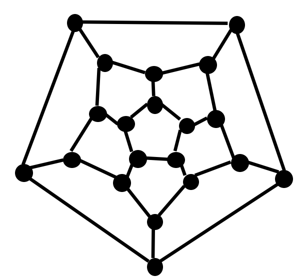
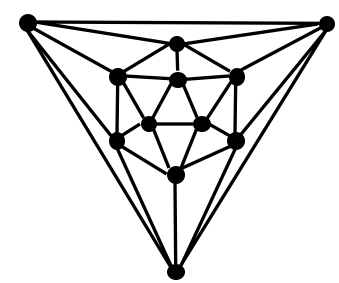
Platonic Solids

- *The simplest regular polytopes*

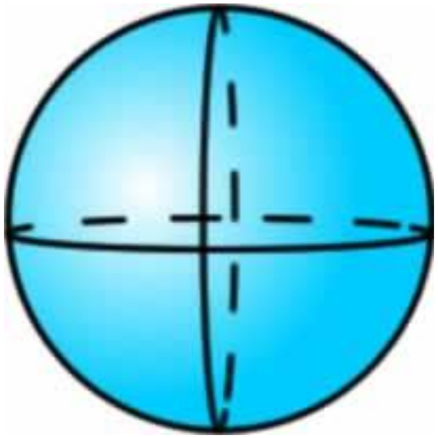
Tetrahedron *Hexahedron* *Octahedron* *Dodecahedron* *Icosahedron*

- *The corresponding graphs*

Euler Characteristics

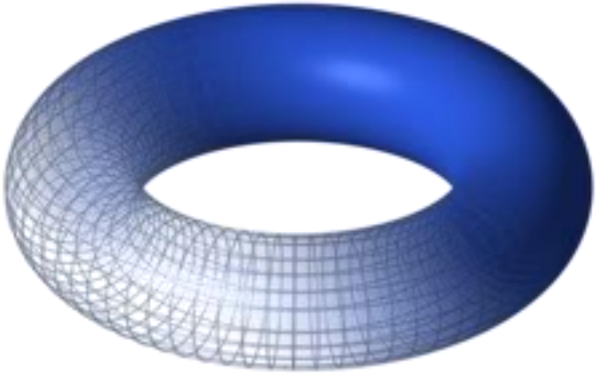
- *Imagine again that a polyhedron is made of rubber. Instead of tearing it, inflate it.*



- *We obtain a sphere with the graph of the polyhedron drawn on it*
- *This graph is planar. Therefore we know that*

$$v - e + r = 2$$

- *2 is the Euler characteristic of a sphere*
- *What if the polyhedron is such that after inflating we obtain a torus?*



$v - e + r = 0$

Non-Planar Graphs

- **Corollary.**

  Let G be a connected planar graph with v vertices, e edges, and r regions. Then $e \le 3v - 6$.

- **Proof.**

  Clearly, the boundary of every region contains at least 3 edges. Let these numbers be $l_1, l_2, \ldots, l_r$.

  Go around every region. The total number of edges we use is

$$l_1 + l_2 + \cdots + l_r \ge 3r.$$

  On the other hand, we have traveled along every edge at most twice. Therefore, $3r \le 2e$. By Euler's formula

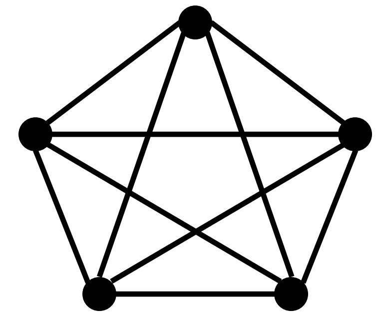
$$3v - 3e + 3r = 6$$

$$3e - 3v + 6 = 3r \le 2e$$

$$3e - 3v + 6 \le 2e \Rightarrow e \le 3v - 6$$

Non-Planar Graphs (cntd)

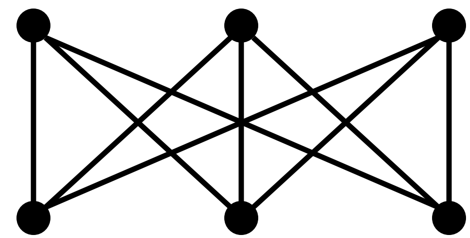
- *The graph K_5 is not planar.*



K_5

- *Indeed, K_5 has $v = 5$ vertices and $e = 10$ edges.*

  $e = 10 > 9 = 3v - 6$



$K_{3,3}$

- *Graph $K_{3,3}$ is not planar.*

- *Note that $K_{3,3}$ is bipartite. Therefore every cycle has length at least 4.*

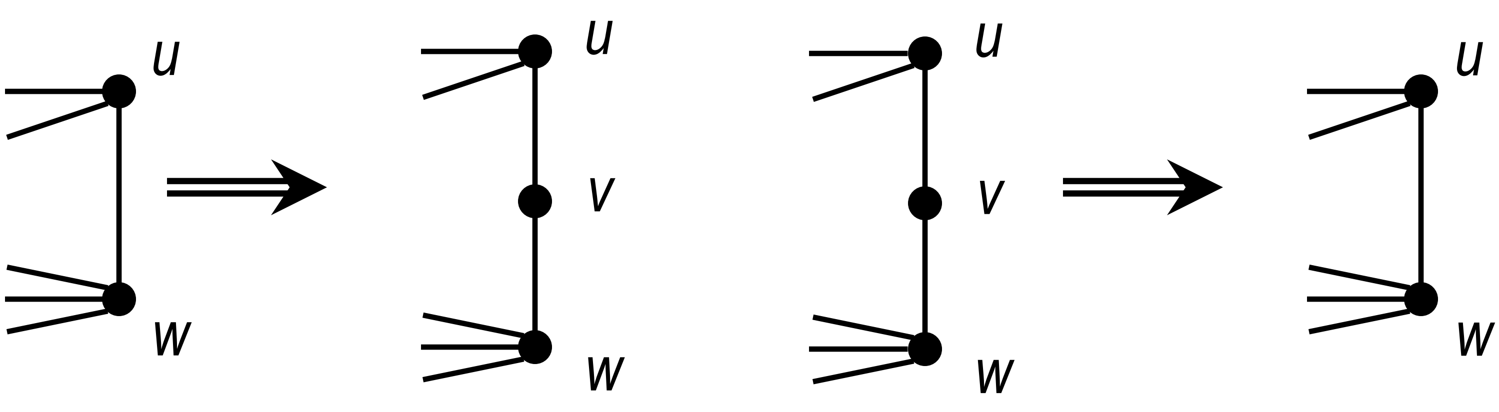
  As in the corollary above we get the inequality $4r \leq 2e$

  By Euler's formula, $r = e - v + 2 = 9 - 6 + 2 = 5$.

  However, $4r = 20 > 18 = 2e$

Subdivisions and Exclusions

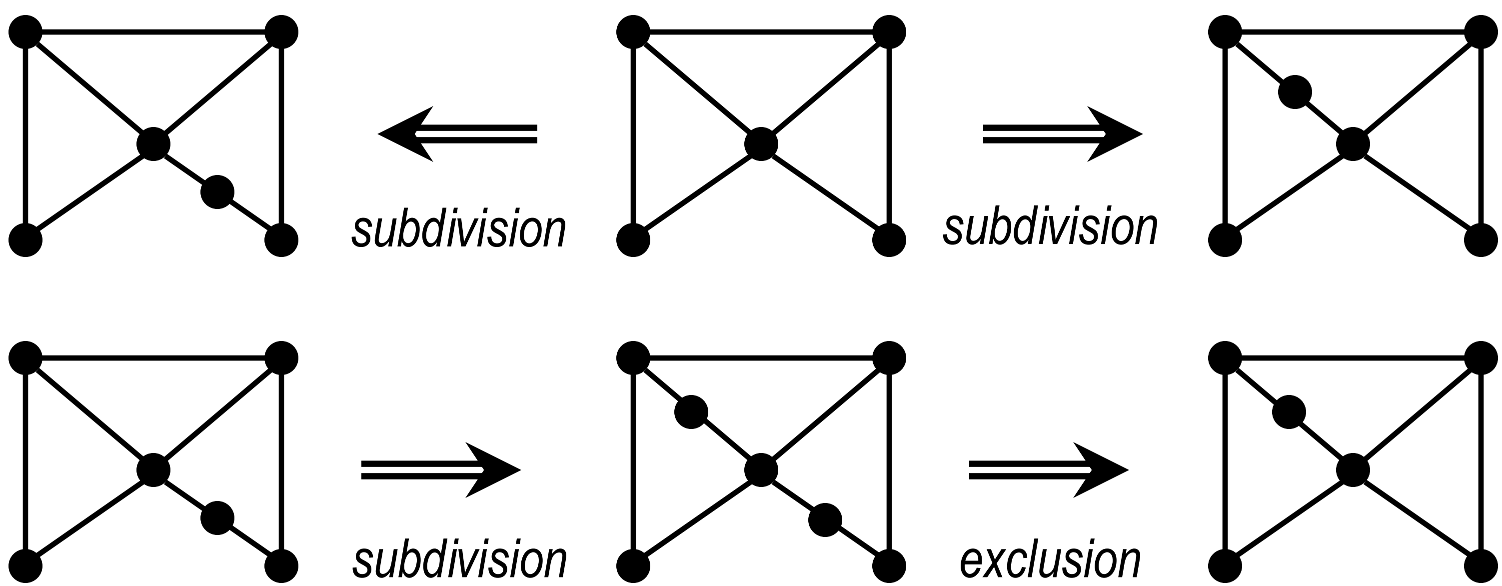
- *Let $G = (V, E)$ be a simple graph. An* *elementary subdivision* *of G results when an edge $e = uw$ is removed from G and then edges uv and vw are added to $G - e$, where $v \notin V$.*
- *Let $v \in V$ be a vertex of degree 2 in G, and let $uv, vw \in E$, where u and w are not adjacent. An* *exclusion* *of G results when the vertex v and edges uv and vw are removed from G and then the edge uw is added.*



elementary subdivision *exclusion*

Homeomorphic Graphs

- *Simple graphs G and H are called homeomorphic if they are isomorphic or if they can both be obtained from the same simple graph by a sequence of elementary subdivisions*
- *Equivalently, the graphs G and H are called homeomorphic if they are isomorphic or if G can be obtained from H by a sequence of elementary subdivisions and exclusions.*



Kuratowski's Theorem

- **Kuratowski's Theorem.**

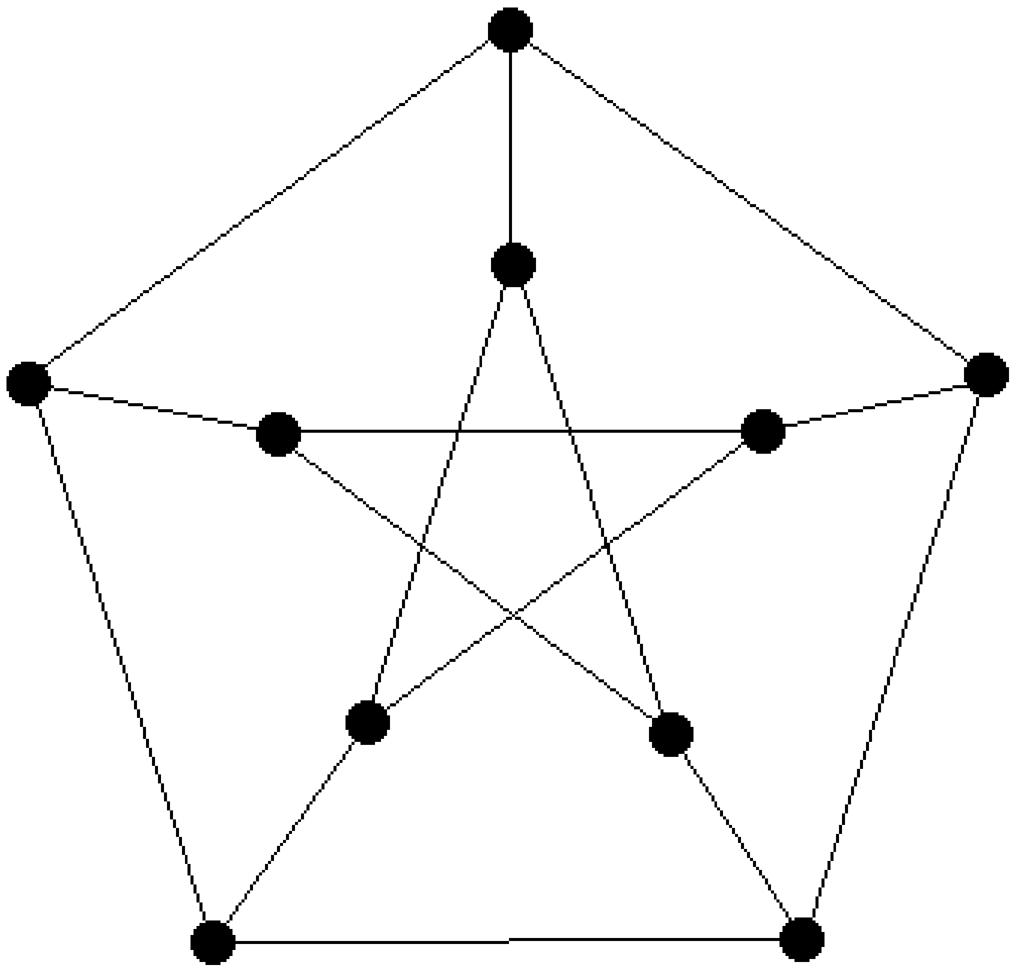
  A graph is nonplanar if and only if it contains a subgraph that is homeomorphic to either K_5 *or* $K_{3,3}$

- **Proof***.*

  One direction is easy: If a subgraph is nonplanar then the entire graph is nonplanar. Also, elementary subdivisions and exclusions do not destroy planarity.

  The other direction is hard.

Petersen Graph



Practice

Exercises from the Book:

7th edition: 1, 13, 19, 22 (page 725 – 726)

8th edition: 1, 13, 19, 22 (page 760 – 761)