MACM 101 — Discrete Mathematics I

Outline Solutions to Exercises on Propositional Logic

1. **There are two tribes living on the island of Knights and Knaves: knights and knaves. You meet three inhabitants of the island of Knights and Knaves, A, B, and C. Let p represents the statement "A is a knight", q represents the statement "B is a knight", and r represents the statement "C is a knight". Use these names along with logic connectives to write each of the following English sentences in symbolic logic notation:**

   (a) **A is a knave and B or C is a knight.**

   (b) **A and B are knaves, or A and C are knights.**

   (c) **At least two people are knights.**

   (d) **Exactly two people are knights.**

   (a) $\neg p \wedge (q \vee r)$.

   (b) $(\neg p \wedge \neg q) \vee (p \wedge r)$.

   (c) $(p \wedge q) \vee (q \wedge r) \vee (r \wedge p)$.

   (d) $(p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r)$.

2. **Find a compound statement involving the propositional variables p, q, and r that is true when p and q are true, and r is false, but false otherwise.**

   $p \wedge q \wedge \neg r$.

3. **There are two tribes living on the island of Knights and Knaves: knights and knaves. Knights always tell truth and knaves always lie. You meet three inhabitants of the island of Knights and Knaves, A, B, and C. A says "B and C are both knaves", B says "Only one of the other two is a knave", and C says "At least one of us is a knave". Who if anyone is a knight?**

   We construct the truth table for each of the statements, then we encode the fact that each statement is true if and only if the speaker is a knight, and then see when all of the equivalence statements are true statements are true. Let p represents the statement "A is a knight", q represents the statement "B is a knight", and r represents the statement "C is a knight". Then the statements are: $\neg q \wedge \neg r$, $p \oplus r$, $\neg p \vee \neg q \vee \neg r$. The equivalence statemnents in that case are $p \leftrightarrow (\neg q \wedge \neg r)$, $q \leftrightarrow (p \oplus r)$, $r \leftrightarrow (\neg p \vee \neg q \vee \neg r)$.

p	q	r	$\neg q \wedge \neg r$	$p \oplus r$	$\neg p \vee \neg q \vee \neg r$	$p \leftrightarrow (\neg q \wedge \neg r)$	$q \leftrightarrow (p \oplus r)$	$r \leftrightarrow (\neg p \vee \neg q \vee \neg r)$
0	0	0	1	0	1	0	1	0
0	0	1	0	1	1	1	0	1
0	1	0	0	0	1	1	0	0
0	1	1	0	1	1	1	1	1
1	0	0	1	1	1	1	0	0
1	0	1	0	0	1	0	1	1
1	1	0	0	1	1	0	1	0
1	1	1	0	0	0	0	0	0

   From the table the only collection of truth values that makes all the equivalence statements true is $p = 0$, $q = r = 1$, that is, A is a knave and B, C are knights.

4. **Construct a truth table for the following compound statement: $(p \leftrightarrow q) \rightarrow (q \oplus \neg r)$.**

p	q	r	$(p \leftrightarrow q) \rightarrow (q \oplus \neg r)$
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

5. Are these system specifications consistent? By consistent we mean **not** a contradiction

   – Whenever the system software is being upgraded, users cannot access the file system.
   – If users can access the file system, then they can save new files.
   – If users cannot save new files, then the system software is not being upgraded.

   Let p represent the statement "the system software is being upgraded", q represent the statements "users can access the file system", and r represent the statement "users can save new files". Then the specifications above are represented by:
   $p \rightarrow \neg q$, $q \rightarrow r$, and $\neg r \rightarrow \neg p$.
   Then we construct the truth tables of these statements and show that they can be all true at the same time.

p	q	r	$p \rightarrow \neg q$	$q \rightarrow r$	$\neg r \rightarrow \neg p$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	1
0	1	1	1	1	1
1	0	0	1	1	0
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	0	1	1

6. **Show that the following compound statement is a contradiction**

$$((q \vee \neg p) \rightarrow (r \wedge p)) \wedge \neg(r \vee \neg q).$$

   Method 1. Construct a truth table.

   Method 2. Use logical equivalences:

$((q \vee \neg p) \rightarrow (r \wedge p)) \wedge \neg(r \vee \neg q)$

$\Longleftrightarrow$	$((q \vee \neg p) \rightarrow (r \wedge p)) \wedge \neg r \wedge q$	DeMorgan's law + double negation
$\Longleftrightarrow$	$(\neg(q \vee \neg p) \vee (r \wedge p)) \wedge \neg r \wedge q$	expression for implication
$\Longleftrightarrow$	$((\neg q \wedge p) \vee (r \wedge p)) \wedge \neg r \wedge q$	DeMorgan's law + double negation
$\Longleftrightarrow$	$((\neg q \vee r) \wedge p) \wedge \neg r \wedge q$	distributive law
$\Longleftrightarrow$	$(\neg q \vee r) \wedge p \wedge \neg r \wedge q$	associative law
$\Longleftrightarrow$	$((\neg q \wedge q) \vee (r \wedge q)) \wedge p \wedge \neg r$	distributive law
$\Longleftrightarrow$	$(r \wedge q) \wedge p \wedge \neg r$	law of contradiction + identity law
$\Longleftrightarrow$	$r \wedge \neg r \wedge p \wedge q$	associativity and commutativity laws
$\Longleftrightarrow$	F	domination law

Method 3. Prove that the formula never takes truth value 1. The formula is a conjunction, therefore to make it true all conjuncts must be true. In particular, $\neg(r \vee \neg q) = 1$, which means $r = 0$ and $q = 1$. With this assignment the premise $q \vee \neg p$ of the first conjunct $(q \vee \neg p) \to (r \wedge p)$ is true, while the conclusion $r \wedge p$ is false. Therefore, twhenever the second conjunct is true the first one is false, implying the statement is never true.

7. **Use truth tables to check if each of the given pairs of compound statements are equivalent:**

   (a) $p \leftrightarrow q$ **and** $(p \to q) \wedge (q \to p)$**.**

   (b) $(\neg q \wedge p) \vee (\neg p \wedge q)$ **and** $\neg p \vee \neg q$**.**

   (c) $(p \wedge q) \vee (q \wedge r) \vee (r \wedge p)$ **and** $(p \vee q) \wedge (q \vee r) \wedge (r \vee p)$**.**

   (a) Equivalent. Straightforward.

   (b) Not equivalent, values are different when $p = q = 0$.

   (c) Equivalent. Straightforward.

8. **How many of the disjunctions** $p \vee \neg q$**,** $\neg p \vee q$**,** $q \vee r$**,** $q \vee \neg r$**, and** $\neg q \vee \neg r$ **can be made simultaneously true by an assignment of truth values to** p**,** q**, and** r**?**

   All of them can be made true simultaneously by setting $p = q = 1$ and $r = 0$.

9. **Use truth tables to determine whether the formula** $(p \wedge \neg q) \to (p \wedge q)$ **is true whenever** $\neg p$ **is true.**

   Straightforward.