MACM 101 — Discrete Mathematics I

Outline Solutions to Exercises on Graphs

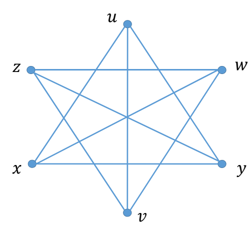
1. **Let $G = (V, E)$ be a simple graph with $|V| = n$. If $|E| > \left(\frac{n}{2}\right)^2$, prove that G cannot be bipartite.**

   We find the maximal number of edges a bipartite graph can have. Suppose that G is bipartite and V_1, V_2, $V_1 \cup V_2 = V$, is the bipartition. Then clearly the maximal number of edges G can have is $|V_1| \cdot |V_2|$ that is reached if we connect every vertex from V_1 to every vertex in V_2. Suppose $|V_1| \leq |V_2|$ and let $a = \frac{n}{2} - |V_1|$. Then
   $$|V_1| \cdot |V_2| = \left(\frac{n}{2} - a\right)\left(\frac{n}{2} + a\right) = \left(\frac{n}{2}\right)^2 - a^2 \leq \left(\frac{n}{2}\right)^2.$$
   Note that the last computation is unnecessary if you are aware of the inequality $mn \leq \left(\frac{m+n}{2}\right)^2$.

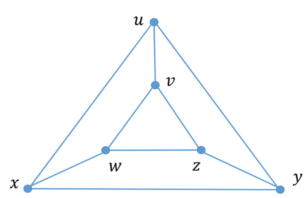
2. **Let $G = (V, E)$ be a simple connected 4-regular planar graph. If $|E| = 16$, how many regions are there in a planar representation of G?**

   By the Handshaking Lemma the number of edges in a graph is half of the sum of the degrees of all the vertices. As G is 4-regular, the degree of each vertex is 4, hence, G has 8 vertices. By Euler's formula $r = e - v + 2 = 16 - 8 + 2 = 10$.

3. **Is the following graph planar? If yes, find the number of regions in its planar representation? If no, what is its Euler characteristic? Explain your answer.**

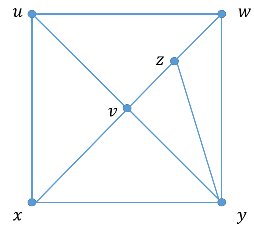
   

   The graph is planar as the following planar representation indicates

   The number of regions is 5, which can either be seen from the picture above, or found by Euler's formula $r = e - v - 2 = 9 - 6 + 2 = 5$.

4. **Find an adjacency list, adjacency matrix, and an incidence matrix of the following graph**



The adjacency list is

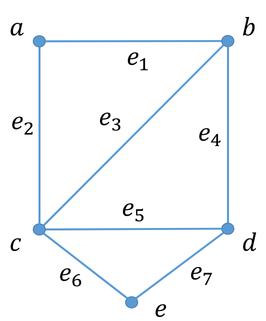
$$
\begin{array}{c||l}
u & x, v, w \\
v & u, z, y, x \\
w & u, z, y \\
x & u, v, y \\
y & x, v, z, w \\
z & v, y, w
\end{array}
$$

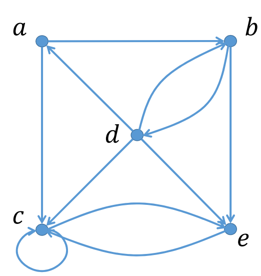
And the matrices are

$$
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & \\
1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0
\end{pmatrix}
\qquad
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.
$$

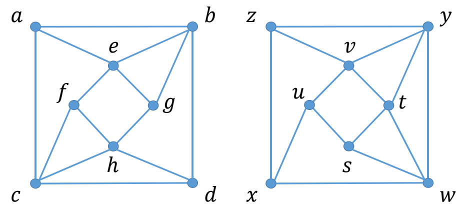
5. **Draw (multi-) (di-)graphs given by the following adjacency and incidence matrices**

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
\qquad
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}.
$$





6. **Are the following graphs isomorphic? Explain.**



This graphs are not isomorphic. Any two vertices of degree 4 of the graph on the right are connected with a path consisting of vertices of degree 4. This is however not the case in the graph on the left.

7. **Let $G = (V, E)$ be an undirected graph. Define a binary relation R on V by $(u, v) \in R$ if and only if $u = v$ or there is a path from u to v. Prove that R is an equivalence relation. What are its equivalence classes?**

   Reflexivity is straightforward, because $(u, v) \in R$ whenever $u = v$, by definition.
   Symmetricity is also straightforward. Since G is undirected, if there is a path from u to v then there is also a path from v to u obtained by tracing the path from u to v backward.
   Transitivity: If there are paths from u to v and from v to w, concatenating these paths we obtain a walk from u to w. Note that this walk is not necessarily a path because some vertices or edges can be reused. However, by one of the properties of connectivity if there is a walk from u to w, there is also a path connecting the two vertices.

   As is easily seen, the equivalence classes of this relation are the connected components of G.

8. **Show that every connected graph with n vertices has at least $n - 1$ edges.**

   We prove the claim by strong induction. The base case, $n = 2$, is obvious, as we need at least 1 edge to connect 2 vertices. For the induction hypothesis suppose that every connected graph with $\ell \leq k$ vertices has at least $\ell - 1$ edges. Take a graph $G = (V, E)$ with $k + 1$ vertices and remove a vertex $v \in V$ from G. The resulting graph $G' = (V - \{v\}, E')$ is not necessarily connected. Let $G_1 = (V_1, E_1), \dots, G_s = (V_s, E_s)$ be the connected components of G'. By the induction hypothesis $|E_i| \geq |V_i| - 1$. Then we have

$$\begin{aligned} |E| &= |E_1| + \cdots + |E_\ell| + deg(v) \geq (|V_1| - 1) + \cdots + (|V_s| - 1) + deg(v) \\ &= (|V_1| + \cdots + |V_s|) + (deg(v) - s) = (|V| - 1) + (deg(v) - s) \geq |V| - 1. \end{aligned}$$

   The last inequality is true because removing a vertex of degree d can produce at most d connected components.

9. **Let $G = (V, E)$ be a simple connected undirected graph, where $V = \{v_1, v_2, v_3, \dots, v_n\}$, $n \geq 2$, $deg(v_1) = 1$, and $deg(v_i) \geq 2$ for $2 \leq i \leq n$. Prove that G has a cycle.**

   Find the sum of the degrees of all the vertices of G

$$\sum_{i=1}^{n} deg(v_i) \geq 1 + \sum_{i=2}^{n} 2 = 2n - 1.$$

   By the Handshaking Lemma $|E| \geq \frac{2n-1}{2} = n - \frac{1}{2}$, that is, $|E| \geq n$. Therefore G is not a tree. As it is connected, it contains a cycle.

10. **Let $T = (V, E)$ be a tree where $|V| = n$. Suppose that for each $v \in V$ either $deg(v) = 1$ or $deg(v) \geq m$, where m is a fixed positive integer, $m \geq 2$, and there is a vertex of degree at least m. What is the smallest value possible for n.**

The smallest possible value for n is $m+1$, Since there is a vertex of degree m, we have $|V| \geq m+1$. The graph in the figure below provides an example of such a graph with exactly $m+1$ vertices.