MACM 101 — Discrete Mathematics I

Outline Solutions to Exercises on Propositional Logic II

1. **Show that $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent. Do not use truth tables.**

   Method 1. It is sufficient to find one assignment of values to p, q, r such that the two statements get different truth values. For instance if $p = q = r = 0$ then $(p \to q) \to r = 0$ while $p \to (q \to r) = 1$.

   Method 2. Use logical equivalences to simplify the biconditional

   $$((p \to q) \to r) \leftrightarrow (p \to (q \to r)).$$

   It can be shown to be equivalent to $p \vee r$, that is not a tautology.

2. **Simplify the compound statement**

   $$((((p \wedge q) \wedge r) \vee ((p \wedge q) \wedge \neg r)) \vee \neg q) \to s.$$

   Use logic equivalences:

   $((((p \wedge q) \wedge r) \vee ((p \wedge q) \wedge \neg r)) \vee \neg q) \to s$

$\iff$	$(((p \wedge q) \wedge (r \vee \neg r)) \vee \neg q) \to s$	distributive law
$\iff$	$((p \wedge q) \vee \neg q) \to s$	law of excluded middle + domination law
$\iff$	$((p \vee \neg q) \wedge (q \vee \neg q)) \to s$	distributive law
$\iff$	$(p \vee \neg q) \to s.$	law of excluded middle + domination law

3. **Are the following compound statements equivalent? Do not use truth tables.**

   $$(s \to (\neg p \vee r)) \wedge ((p \to (r \vee q)) \wedge s), \quad \textbf{and} \quad s \wedge q.$$

   Use logic equivalences to try to show that these statements are equivalent. (**Note!** that this way you cannot prove that the statements are not equivalent!)

   $(s \to (\neg p \vee r)) \wedge ((p \to (r \vee q)) \wedge s)$

$\iff$	$(\neg s \vee \neg p \vee r) \wedge ((\neg p \vee r \vee q) \wedge s)$	expression for implication
$\iff$	$(\neg s \vee \neg p \vee r) \wedge (\neg p \vee r \vee q) \wedge s$	associativity law
$\iff$	$((\neg s \wedge s) \vee (\neg p \vee r) \wedge s) \wedge (\neg p \vee r \vee q)$	distributivity law
$\iff$	$(\neg p \vee r) \wedge s \wedge (\neg p \vee r \vee q)$	law of contradiction, identity law
$\iff$	$(\neg p \vee r) \wedge s$	absorption law

   We now see that after simplification the two compound statements look quite different. So, we suspect they are not. This, however, does not prove they are not equivalent. To prove that we need to find a truth assignment that makes one of them true and the other one false. Fortunately, it is quite easy, as q does not appear in the first statement. We set $q = 0$, thus making $s \wedge q$ false, and then select any assignment that makes $(\neg p \vee r) \wedge s$ true, for instance, $p = r = s = 1$.

4. **Let NAND be the logic connective defined by $p \uparrow q \Leftrightarrow \neg(p \wedge q)$. Show that $p \uparrow (q \uparrow r)$ is not logically equivalent to $(p \uparrow q) \uparrow r$.**

   Method 1. Construct truth tables.

   Method 2. We try to find a truth assignment that separates the two statements. Observe that $p \uparrow (q \uparrow r)$ is false only if $p = 1$ and $q \uparrow r$ is true. On the other hand, $(p \uparrow q) \uparrow r$ is true whenever $r = 0$. It only remains to find q such that $q \uparrow r$ is true assuming $r = 0$. Any truth value of q will do, take $q = 0$. Thus for $p = 1, q = r = 0$ the statement $p \uparrow (q \uparrow r)$ is false while $(p \uparrow q) \uparrow r$ is true.

5. **Prove that the Rule of Proof by Cases is a valid argument.**

It suffices to prove that the corresponding expression

$$((p \to r) \land (q \to r)) \to ((p \lor q) \to r)$$

is a tautology.

Method 1. Construct the truth table.

Method 2. Use logical equivalences to show that this statement is equivalent to 1.

$((p \to r) \land (q \to r)) \to ((p \lor q) \to r)$	
$\iff \neg((p \to r) \land (q \to r)) \lor ((p \lor q) \to r)$	expression for implication
$\iff \neg((\neg p \lor r) \land (\neg q \lor r)) \lor (\neg(p \lor q) \lor r)$	expression for implication
$\iff (\neg(\neg p \lor r) \lor \neg(\neg q \lor r)) \lor (\neg(p \lor q) \lor r)$	De Morgan's law
$\iff (p \land \neg r) \lor (q \land \neg r) \lor \neg(p \lor q) \lor r$	De Morgan's law + double negation law
$\iff ((p \lor q) \land \neg r) \lor \neg(p \lor q) \lor r$	distributive law
$\iff ((p \lor q \lor r) \land (\neg r \lor r)) \lor \neg(p \lor q)$	distributive law
$\iff (p \lor q \lor r) \lor \neg(p \lor q)$	law of excluded middle + domination law
$\iff (p \lor q) \lor r \lor \neg(p \lor q)$	associative law
$\iff T \lor r$	law of excluded middle
$\iff$ T	domination law

Method 3. Suppose that the premises $(p \to r)$ and $(q \to r)$ are true. If $p \lor q$ is false then, by definition of implication, the conclusion $(p \lor q) \to r$ is true. Therefore let $p \lor q$ be true. From $p \lor q = 1$ we know that either p or q is true. Suppose p is true. Then, since $p \to r$ is true, by definition of implication, r must be true. If q is true then, as $q \to r$ is true, we again conclude that r must be true. Thus, in all cases r is true, and therefore the conclusion is true. Q.E.D.

Method 4. Show that starting from premises $p \to r$ and $q \to r$ one can infer the conclusion $(p \lor q) \to r$:

Steps	Reason
1. $p \to r$	premise
2. $\neg p \lor r$	expression for implication to Step 1
3. $q \to r$	premise
4. $\neg q \lor r$	expression for implication to Step 3
5. $(\neg p \lor r) \land (\neg q \lor r)$	rule of conjunction to Steps 2 and 4 (see Exercise 8)
6. $(\neg p \land \neg q) \lor r$	DeMorgan's law to Step 5
7. $\neg(p \lor q) \lor r$	DeMorgan's law to Step 6
8. $(p \lor q) \to r$	expression for implication to Step 7

6. **Each of two rooms (room I and room II) contains either a lady or a tiger. If a room contains a lady, the sign on its door is true. If it contains a tiger, the sign is false. The signs are**

I	**II**
AT LEAST ONE ROOM CONTAINS A LADY	**THE OTHER ROOM CONTAINS A LADY**

Which rooms contain ladies?

Let the primitive statements be:

p, 'the first room contains a lady'
q, 'the second room contain a lady'

Then the sign on the first door says that $p \lor q$, and the sign on the second door claims p. We know that a sign is true if and only if the corresponding room contains a lady. Therefore the statements $p \leftrightarrow (p \vee q)$ and $p \leftrightarrow q$ are true.

By checking all possible truth values of p, q we find that both statements above are true only if $p = q = 1$ or $p = q = 0$, that is, both rooms contain ladies, or both rooms contain tigers.

7. **What relevant conclusion or conclusions can be drawn from this set of premises? Explain the rules of inference used to obtain each conclusion from the premises.**
"If I eat spicy foods, then I have strange dreams."
"I have strange dreams if there is thunder while I sleep."
"I did not have strange dreams."

The relevant conclusions are: "I did not eat spicy food" and "There is no thunder while I sleep".

Let the primitive statements be:

s, 'I eat spicy foods'
d, 'I have strange dreams'
t, 'There is thunder while I sleep'

Then the premises are translated as: $s \to d$, $t \to d$, and $\neg d$.

And the conclusions: $\neg s$, $\neg t$.

Steps	Reason
1. $s \to d$	premise
2. $\neg d$	premise
3. $\neg s$	Modus Tollens to Steps 1 and 2
4. $t \to d$	premise
5. $\neg t$	Modus Tollens to Steps 4 and 2.

8. **Write the following argument in symbolic form. Then establish the validity of the argument or give a counterexample to show that it is invalid.**

> **If Dominic goes to the racetrack, then Helen will be mad. If Ralph plays cards all night, then Carmela will be mad. If either Helen or Carmela gets mad, then Veronica (their attorney) will be notified. Veronica has not heard from either of these two clients. Consequently, Dominic didn't make it to the racetrack and Ralph didn't play cards all night.**

Let the primitive statements be:

d, 'Dominic goes to the racetrack'
h, 'Helen is mad'
r, 'Ralph plays cards all night'
c, 'Carmela is mad'
v, 'Veronica is notified'

Then the premises are translated as: $d \to h$, $r \to c$, $(h \vee c) \to v$, and $\neg v$.

And the conclusions: $\neg d$, $\neg r$.

Steps	Reason
1. $(h \vee c) \to v$	premise
2. $\neg v$	premise
3. $\neg(h \vee c)$	Modus Tollens to 1 and 2
4. $\neg h \wedge \neg c$	DeMorgan's law to 3
5. $\neg h$	rule of simplification to 4
6. $d \to h$	premise
7. $\neg d$	Modus Tollens to 5 and 6

8. $\neg c$	rule of simplification to 4
9. $r \to c$	premise
10. $\neg r$	Modus Tollens to 8 and 9

Using the rule of conjunction from the next problem we can also obtain $\neg d \wedge \neg r$.

9. **Write the following argument in symbolic form. Then establish the validity of the argument or give a counterexample to show that it is invalid.**

> **If Newton is not considered a great mathematician and Leibniz's work is not ignored, then calculus would not be the centerpiece of the modern math curriculum. Newton is considered the greatest mathematician only if Leibniz's work is ignored. Therefore, calculus is the centerpiece of the modern math curriculum and Leibniz's work is not ignored.**

Let the primitive statements be:

n, 'Newton is considered a great mathematician'
ℓ, 'Leibniz's work is ignored'
c, 'calculus is the centerpiece of the modern math curriculum'

Then the premises are translated as: $(\neg n \wedge \neg \ell) \to \neg c$, $n \to \ell$.

And the conclusion: $c \wedge \neg \ell$.

Now notice that if $n = 0, \ell = 1, c = 0$, then both premises are true, while the conclusion is false. This means that the argument is invalid.

10. **Using rules of inference and logic equivalences give the reasons for the steps verifying the following argument.**
Premises: $(t \vee q) \to (p \wedge \neg r)$**,** $(u \vee \neg s) \to t$**,** $(p \wedge \neg r) \to q$**,** $\neg s \vee t \vee q$**,** $\neg t \wedge u$**,** $p \vee \neg q$**.**
Conclusion: p**.**

Steps	Reasons
1) $\neg t \wedge u$	premise
2) $\neg t$	rule of simplification
3) $(u \vee \neg s) \to t$	premise
4) $\neg t \to \neg(u \vee \neg s)$	contrapositive
5) $\neg t \to (\neg u \wedge s)$	De Morgan's law
6) $\neg u \wedge s$	Modus Ponens to 2 and 5
7) s	rule of simplification
8) $(t \vee q) \to (p \wedge \neg r)$	premise
9) $\neg s \vee t \vee q$	premise
10) $s \to (t \vee q)$	expression for implication
11) $s \to (p \wedge \neg r)$	rule of syllogism to 10 and 8
12) $p \wedge \neg r$	Modus Ponens to 7 and 11
13) $(p \wedge \neg r) \to q$	premise
14) q	Modus Ponens to 12 and 13
15) $p \vee \neg q$	premise
16) p	rule of disjunctive syllogism (or rule of resolution)