# MACM 101 — Discrete Mathematics I

## Outline Solutions to Exercises on Predicates and Quantifiers

1. **Write each of the following statements using quantifiers and predicates. In each case, you must specify the domain and define the predicates you use.**

   **(a) There are computer science majors who do not minor in mathematics.**

   **(b) Every two real numbers have an integer in between.**

   **(c) No square number immediately follow a prime number.**

   (a) Let the domain be the set of all people in the world. Let $C(x)$ denote that $x$ is a major in computer science, and let $M(x)$ denote that $x$ is a minor in mathematics. Then the statement is

   $$\exists x(C(x) \wedge \neg M(x)).$$

   (b) Let the domain be the set of all real numbers. Let $I(x)$ denote that $x$ is an integer, and let $x < y$ have the usual meaning. Then the statemnt is

   $$\forall x, y \exists z(I(z) \wedge (x < z) \wedge (z < y)).$$

   (c) Let the domain be the set of all positive integers. Let $P(x)$ denote that $x$ is a prime number, let $S(x)$ denote that $x$ is a square number, and let $N(x, y)$ means that $y = x + 1$. Then the statemnt is

   $$\neg\exists x, y(P(x) \wedge S(y) \wedge N(x, y)) \quad \text{or} \quad \forall x, y((P(x) \wedge S(y)) \rightarrow \neg N(x, y)).$$

   Alternatively, we may use $y = x^2$ to replace $S(y)$ and $y = x + 1$ to replace $N(x, y)$. Then we have

   $$\neg\exists x, y(P(x) \wedge (x + 1 = y^2)) \quad \text{or} \quad \forall x, y(P(x) \rightarrow \neg(x + 1 = y^2)),$$

   although this is not a purely logic formula, and so is not quite satisfactory in the context of just logic of predicates.

2. **Let $F(x, y)$ be the statement "$x$ can fool $y$", where the domain consists of all people in the world. Use quantifiers to express each of these statements.**

   **(a) Everybody can fool somebody.**

   **(b) There is no one who can fool everybody.**

   **(c) Everyone can be fooled by somebody.**

   **(d) No one can fool both Fred and Jerry.**

   **(e) Nancy can fool exactly two people.**

   (a) $\forall x \exists y F(x, y)$.

   (b) $\neg(\exists x \forall y F(x, y)$.

   (c) $\forall x, \exists y F(y, x)$

   (d) $\neg(\exists x(F(x, \text{Fred}) \wedge F(x, \text{Jerry}))$.

   (e) $\exists x, y((x \neq y) \wedge F(\text{Nancy}, x) \wedge F(\text{Nancy}, y) \wedge (\forall z(F(\text{Nancy}, z) \rightarrow ((z = x) \vee (z = y)))))$.

3. **Let the domain be the set $D = \{1, 3, 5, 7, 8, 10, 11, 12\}$. Write each of the following statements using predicates and quantifiers, and decide whether it is true for all the elements from $D$. If it is not, give a counterexample.**

**(a) If $x$ is even and $x > 7$, then $x < 20$.**

**(b) If $x$ is odd, then $x < 10$.**

**(c) If $x$ is odd or $x < 5$, then $x - 1$ is even.**

Let $E(x)$ denote that $x$ is even. Then $\neg E(x)$ means that $x$ is odd.

(a) $\forall x((E(x) \land (x > 7)) \to (x < 20))$. True.

(b) $\forall x(\neg E(x) \to (x < 10))$. False, $x = 11$ is a counterexample.

(c) $\forall x((\neg E(x) \lor (x < 5)) \to E(x - 1))$. True.

4. **Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.**

   **(a)** $\exists x \, \forall y \, (y \neq 0 \to (xy = 1))$**;**

   **(b)** $\forall x \forall y \exists z \, (z = \frac{x+y}{2})$**.**

   (a) This statement is false. Take any vaue of $x$ then for any $y$ it should be the case that $xy = 1$ provided $y \neq 0$. In particular it must be true for $y = 1$ and $y = 2$. However, it is not possible that $1 \cdot x = 2 \cdot x = 1$.

   (b) This statement is true. Take generic values of $x$ and $y$, say, $c$ and $d$. Then by choosing $z = \frac{c+d}{2}$ we satisfy the requirement of the statement.

5. **Use predicates and quantifiers to express this statement**

   **"Some students in this class grew up in the same town as at least two other students in this class."**

   There are several ways to express this. Here is one of them.
   Let the predicates be:

   $C(x)$, 'student $x$ is in this class',
   $T(x, y)$, 'student $x$ grew up in the same town as student $y$'

   Take a student $x$. The condition that there is another student in this class who grew up in the same town we can express as

   $$\exists y \Big( C(y) \land (x \neq y) \land T(x, y) \Big).$$

   The condition that there at least two such students can be expressed by using the statement above combined with a statement that says that that there are two such student, say, $y$ and $z$ that are different and different from $x$:

   $$\exists y \exists z \big( C(y) \land C(z) \land (x \neq y) \land (x \neq z) \land (y \neq z) \land T(x, y) \land T(x, z) \big).$$

   Finally, we just need to say that such $x$ exists:

   $$\exists x \exists y \exists z \big( C(y) \land C(z) \land (x \neq y) \land (x \neq z) \land (y \neq z) \land T(x, y) \land T(x, z) \big).$$

6. **Find a counterexample, if possible, to this universally quantified statement, where the domain for all variables consists of all integers**

   $$\forall x \exists y \; (2x + 5y \leq x + y).$$

   This statement is true, thus there is no counterexample. Take any integer $x$, we need to find $y$ such that $2x+5y \leq x + y$, that is, $x \leq -4y$. So, taking $y = \frac{x}{-4}$ we satisfy the inequality $2x + 5y \leq x + y$.

7. **Write the negation of each of the following statements using the "if, then" structure.**

   **(a) There exists a positive integer $n$ such that $n$ is even and $\frac{1}{n} > 1$.**

   **(b) There exist positive integers $a$ and $b$ such that $a - b$ is odd and $a^2 = 2b^2$.**

(a) For any positive integer $n$, if $n$ is even, then $\frac{1}{n} \not> 1$.

(b) For any positive integers $a, b$, if $a - b$ is odd, then $a^2 \neq 2b^2$.

8. **Find a domain for variables $x$, $y$, and $z$ for which the statement**

$$\exists x \exists y \forall z ((x = z) \vee (y = z))$$

**is true and another domain in which it is false.**

Observe that the statement in fact claims that the domain contains at most two different elements (there are two elements, not necessarily different, such that any third element equals to one of them). Thus the statement is true in $\{1, 2\}$ and false in $\{1, 2, 3\}$. Indeed, in the first case one can select $x = 1, y = 2$ and then for any element $z$ from $\{1, 2\}$ either $z = 1 = x$ or $z = 2 = y$. For $\{1, 2, 3\}$ we can argue as follows. No matter how we pick $x, y$, there is always a value $c$ in $\{1, 2, 3\}$ such that $c \neq x$, $c \neq y$. This element $c$ provides a counterexample to the innermost universally quantified statement $\forall z ((x = z) \wedge (y = z))$.