MACM 101 — Discrete Mathematics I

Outline Solutions to Exercises on Predicates and Quantifiers

1. **Rewrite the following statement so that negations appear only within predicates (that is, no negation is outside a quantifier or an expression involving logical connectives)**

$$\neg\forall x\ ((\forall y\exists z\ P(x,y,z)) \oplus (\forall z\forall y\ (R(x,y,z) \vee S(z,y)))).$$

There are two methods to approach this problem.

Method 1. Use the observation that $\neg(p \leftrightarrow q) \iff p \oplus q$.

$$\begin{aligned}
&\neg\forall x\ ((\forall y\exists z\ P(x,y,z)) \oplus (\forall z\forall y\ (R(x,y,z) \vee S(z,y))) \\
&\iff \exists x\ \neg((\forall y\exists z\ P(x,y,z)) \oplus (\forall z\forall y\ (R(x,y,z) \vee S(z,y))) && \text{law for negation and quantifiers} \\
&\iff (\forall y\exists z\ P(x,y,z)) \leftrightarrow (\forall z\forall y\ (R(x,y,z) \vee S(z,y))) && \text{observation } \neg(p \oplus q) \iff p \leftrightarrow q
\end{aligned}$$

Method 2. Use logic equivalences

$$\begin{aligned}
&\neg\forall x\ ((\forall y\exists z\ P(x,y,z)) \oplus (\forall z\forall y\ (R(x,y,z) \vee S(z,y))) \\
&\iff \exists x\ \neg((\forall y\exists z\ P(x,y,z)) \oplus (\forall z\forall y\ (R(x,y,z) \vee S(z,y))) \\
& && \text{law for negation and quantifiers} \\
&\iff \exists x\ \neg\Big(\big((\forall y\exists z\ P(x,y,z)) \to \neg(\forall z\forall y\ (R(x,y,z) \vee S(z,y)))\big) \\
&\qquad \wedge\big(\neg(\forall y\exists z\ P(x,y,z)) \to (\forall z\forall y\ (R(x,y,z) \vee S(z,y)))\big)\Big) \\
& && \text{expression for exclusive OR} \\
&\iff \exists x\ \Big(\neg\big((\forall y\exists z\ P(x,y,z)) \to \neg(\forall z\forall y\ (R(x,y,z) \vee S(z,y)))\big) \\
&\qquad \vee\neg\big(\neg(\forall y\exists z\ P(x,y,z)) \to (\forall z\forall y\ (R(x,y,z) \vee S(z,y)))\big)\Big) \\
& && \text{DeMorgan's law} \\
&\iff \exists x\ \Big(\neg\big(\neg(\forall y\exists z\ P(x,y,z)) \vee \neg(\forall z\forall y\ (R(x,y,z) \vee S(z,y)))\big) \\
&\qquad \vee\neg\big((\forall y\exists z\ P(x,y,z)) \vee (\forall z\forall y\ (R(x,y,z) \vee S(z,y)))\big)\Big) \\
& && \text{expression for implication} \\
&\iff \exists x\ \Big(\big((\forall y\exists z\ P(x,y,z)) \wedge (\forall z\forall y\ (R(x,y,z) \vee S(z,y)))\big) \\
&\qquad \vee\big(\neg(\forall y\exists z\ P(x,y,z)) \wedge \neg(\forall z\forall y\ (R(x,y,z) \vee S(z,y)))\big)\Big) \\
& && \text{DeMorgan's law and double negation} \\
&\iff \exists x\ \Big(\big((\forall y\exists z\ P(x,y,z)) \wedge (\forall z\forall y\ (R(x,y,z) \vee S(z,y)))\big) \\
&\qquad \vee\big((\exists y\forall z\ \neg P(x,y,z)) \wedge (\exists z\exists y\ (\neg R(x,y,z) \wedge \neg S(z,y)))\big)\Big) \\
& && \text{DeMorgan's law and law for negation and quantif}
\end{aligned}$$

2. **Show that quantified statements**

$$\forall x(P(x) \oplus Q(x)) \quad \textbf{and} \quad (\forall x P(x)) \oplus (\forall x Q(x))$$

are not logically equivalent.

We need to find a counter-example. Let the universe be the set of all working cars, $P(x)$ means that x has wheels, and $Q(x)$ means that x is red. Then the first statement says that every car that is either has wheels or is red (but not both). This is clearly false. The second statement claims that either all working cars have wheels, or all gars are red. Clearly every working car has wheels, that is, $\forall x P(x)$ is true, but there are working cars that are not red, meaning $\forall x Q(x)$ is false. Thus, the second statement is true.

3. **Prove that quantified statements**

$$\forall x \forall y \exists z P(x,y,z) \quad \textbf{and} \quad \forall x \exists z \forall y P(x,y,z)$$

are not logically equivalent.

We need to find a counter-example. Let the universe for all 3 variables be the set of all integers, $\mathbb{Z}$, and let $P(x,y,z)$ be true if and only if $x+y+z=0$. Then the first statement states that no matter what integers x,y are, one can always find z such that $x+y+z=0$. This is clearly true, because z can be chosen to be $z=-x-y$. The second staments says that for any integer x there is a value of z such that $x+y+z=0$, no matter waht y is. This is obvious ly not true, and a counterexample can be constructed as follows. Choose $x=0$, then for any choice of z the value $x+y+z$ is not 0 either for $y=0$ or foe $y=1$.

4. **Prove that the statements**

$$\forall x \forall y P(x,y) \quad \textbf{and} \quad \forall x \forall y P(y,x)$$

are logically equivalent.

Method 1. Observe that both statements claim that $P(a,b)$ is true for any values a,b from the universe.

Method 2. Use logic inference. We infer $\forall x \forall y P(y,x)$ from $\forall x \forall y P(x,y)$, the reverse inference is analogous. So let $\forall x \forall y P(x,y)$ be a premise.

Steps	Reason
1. $\forall x \forall y P(x,y)$	premise
2. $\forall y P(a,y)$	rule of universal specification to Step 1, where a is a generic element
3. $P(a,b)$	rule of universal specification to Step 2, where b is a generic element
4. $\forall x P(a,x)$	rule of universal generalization
5. $\forall y \forall x P(y,x)$	rule of universal generalization.

5. **Assume that $\exists x \forall y P(x,y)$ is true and that the universe for x,y is nonempty. Which of the following must be true?**

$$\forall x \exists y P(x,y) \quad \textbf{and} \quad \exists x \exists y P(x,y).$$

Prove your answer either using a regular proof in the natural language or by giving a formal inference. In the latter case you may need to use the existential rules of inference.

Method 1. Statement $\exists x \forall y P(x,y)$ means that there is a value a of x such that no matter what $y=b$ is, $P(a,b)$ is true. Let us first show that this implies $\exists x \exists y P(x,y)$. Suppose that a nonempty universe U and an interpretation of P are such that $\exists x \forall y P(x,y)$. Then there is a value $a \in U$ such that $\forall y P(a,y)$ is true. Take any $b \in U$, we have $P(a,b)$ is true. Therefore $\exists x \exists y P(x,y)$ is true. To show that $\forall x \exists y P(x,y)$ does not have to be true, we give a counterexample. Let the universe be the set of natural numbers $\mathbb{N}$ (excluding 0) and $P(x,y)$ is true if and only if $x \cdot y = y$. Then $\exists x \forall y P(x,y)$ is true, because if we choose $x=1$ then $1 \cdot y = y$ for all $y \in \mathbb{N}$. However, it is easy to find a counterexample for $\forall x \exists y P(x,y)$. Set $x=2$ then there is no $y \in \mathbb{N}$ such that $2 \cdot y = y$.

Method 2. Use logic inference to prove that $\exists x \exists y P(x,y)$ must be true whenever $\exists x \forall y P(x,y)$ is true. We infer $\exists x \exists y P(x,y)$ from $\exists x \forall y P(x,y)$. So let $\exists x \forall y P(x,y)$ be a premise.

Steps	Reason
1. $\exists x \forall y P(x,y)$	premise
2. $\forall y P(a,y)$	rule of existential specification to Step 1, where a is an element given by

	the existential quantifier
3. $P(a, b)$	rule of universal specification to Step 2, where b is a generic element
4. $\exists y P(a, y)$	rule of existential generalization
5. $\exists x \exists y P(y, x)$	rule of existential generalization.

6. **Determine whether the following argument is valid or invalid and explain why.**

   'Everyone who eats granola every day is healthy.'
   'Linda is not healthy.'
   'Therefore, Linda does not eat granola every day.'

   Express these statements symbolically. Let the predicates be:

   $G(x)$, 'x eats granola every day',
   $H(x)$, 'x is healthy

   Then the premises are translated as: $\forall x\ (G(x) \to H(x))$, $\neg H(\text{Linda})$.

   And the conclusion: $\neg G(\text{Linda})$.

Steps	Reason
1. $\forall x\ (G(x) \to H(x))$	premise
2. $(G(\text{Linda}) \to H(\text{Linda}))$	rule of universal specification
3. $\neg H(\text{Linda})$	premise
4. $\neg G(\text{Linda})$	modus tollens.

7. **Determine whether the following argument is valid or invalid and explain why by giving a formal inference if the argument is valid, or by explaining why a certain step in the argument is incorrect or providing a counterexample if the argument is invalid.**

   'All discrete mathematics students can tell a valid argument from invalid,'
   'All thoughtful people can tell a valid argument from invalid,'
   'Therefore, all discrete mathematics students are thoughtful'

   The argument is invalid. Suppose there exists a discrete mathematics student A can tell a valid argument from invalid, but not thoughtful, while all other discrete mathematics students can tell a valid argument from invalid, and thoughtful. Also assume that all thoughtful people can tell a valid argument from invalid. Then the first premise holds because every discrete mathematics student, including A can tell a valid argument from invalid. The second premise is also true, just by assumption. However, A provides a counterexample to the conclusion.

8. **Given premises:**

   'All hummingbirds are richly colored.'
   'No large birds live on honey.'
   'Birds that do not live on honey are dull in color.'

   infer the conclusion

   'Hummingbirds are small.'

   Let the predicates be:

   $H(x)$, 'x is a hummingbird',
   $C(x)$, 'x is richly colored,
   $S(x)$, 'x is small',
   $L(x)$, 'x lives on honey'

   Then the premises are translated as: $\forall x\ (H(x) \to R(x))$, $\forall x\ \neg(\neg S(x) \wedge L(x))$, $\forall x\ (\neg L(x) \to \neg R(x))$.

   And the conclusion: $\forall x\ (H(x) \to S(x)$.

Steps	Reason
1. $\forall x\ (H(x) \to R(x))$	premise

2. $H(a) \to R(a)$	rule of universal specification to Step 1, where a is a generic element
3. $\forall x\ (\neg L(x) \to \neg R(x))$	premise
4. $\neg L(a) \to \neg R(a)$	rule of universal specification to Step 3, where a is a generic element
5. $R(a) \to L(a)$	contrapositive to Step 4
6. $H(a) \to L(a)$	rule of syllogism to Steps 2 and 5
7. $\forall x\ \neg(\neg S(x) \wedge L(x))$	premise
8. $\neg(\neg S(a) \wedge L(a))$	rule of universal specification to Step 7, where a is a generic element
9. $L(a) \to S(a)$	DeMorgan's law and expression for implication to Step 8
10. $H(a) \to S(a)$	rule of syllogism to Steps 6 and 9
11. $\forall x\ (H(x) \to S(x))$	rule of universal generalisation to Step 10.

9. **By providing a formal inference justify the rule of universal transitivity:**
   if $\forall x(P(x) \to Q(x))$ and $\forall x(Q(x) \to R(x))$ are true, then $\forall x(P(x) \to R(x))$ is also true.

   Premises: $\forall x(P(x) \to Q(x)), \forall x(Q(x) \to R(x))$
   Conclusion: $\forall x(P(x) \to R(x))$
   Argument:

Steps	Reason
1. $\forall x(P(x) \to Q(x))$	premise
2. $\forall x(Q(x) \to R(x))$	premise
3. $P(a) \to Q(a)$	rule of universal specification to Step 1, where a is a generic element
4. $Q(a) \to R(a)$	rule of universal specification to Step 2, where a is a generic element
5. $P(a) \to R(a)$	rule of syllogism to Steps 3 and 4
6. $\forall x\ (P(x) \to R(x))$	rule of universal generalisation to Step 5.

10. **What is wrong with this proof?**

    Theorem. 7 is divisible by 3

    Proof. Every integer number is divisible by 3 or it is not. Let c be an arbitrary integer number. Therefore, it is divisible by 3 or it is not. Suppose it is divisible by 3. By the rule of universal generalization, if an arbitrary number is divisible by 3, every number is is divisible by 3. Therefore, 7 is divisible by 3.

    Express these statements symbolically. Let the predicates be:

    $D(x)$, 'x is divisible by 3',
    $N(x)$, 'x is not divisible by 3'

    Then the theorem translated as: $D(7)$.

    Express the proof in formal terms. First we start off with the statement $\forall x\ (D(x) \vee N(x))$, which is true in the universe of integers. Then we use the rule of universal specification to obtain $D(n) \vee N(n)$ for a generic value n of the variable. Next we say that since the disjunction is true, one of the conditions is true, and assume that the first one is true. Having $D(n)$ we use the rule of universal generalization to conclude $\forall x\ D(x)$, and then the rule of universal specification to obtain $D(7)$.

    Thus, the premise used: $\forall x\ (D(x) \vee N(x))$ (the true statement in the proof)

    And the conclusion: $D(n)$.
    Then the argument may look as follows:

Steps	Reason
1. $\forall x\ (D(x) \vee N(x))$	premise
2. $(D(n) \vee N(n))$	rule of universal specification
3. $D(n)$	some rule of inference
4. $\forall x D(x)$	rule of universal generalization
5. $D(7)$	rule of universal specification.

Note that the rule of inference used in Step 3 is not known to us, and it looks like

$$\frac{p \vee q}{\triangle \;\; p}$$

If it is a valid rule of inference, the following statement must be a tautology

$$(p \vee q) \rightarrow p.$$

However, it is not a tautology, since when $p = 0, q = 1$ the statement is false. Thus the argument is invalid.

Less formally, the mistake made in the proof is a mishandling the generic value n. For a generic value we cannot assume any property that is not shared by every element in the universe. Therefore $D(n) \vee N(n)$ is a property we can assume for the generic value n, while $D(n)$ is not. We can consider cases proving some statements. For instance, we want to prove some statement $Q(n)$ for a generic value n. Then we can first assume n is divisible by 3, and prove $Q(n)$ in this case, then assume n is not divisible by 3 and prove $Q(n)$ in this case. Note that the rule of universal generalization we can only apply when $Q(n)$ is proved in both cases. This means $Q(n)$ is true for any element in the universe, no matter, if it is divisible by 3 or not.

11. Prove that the sum of an irrational number and a rational number is irrational.

We prove by contradiction. First assume that there exist numbers a, b such that a is irrational and b and $a + b$ are both rational. Then $b = \frac{c}{d}$ and $a + b = \frac{e}{f}$ by definition. But

$$a = (a + b) - b = \frac{e}{f} - \frac{c}{d} = \frac{de - fc}{fd}$$

and hence a is also rational. Therefore, we have a contradiction and it follows that an irrational and a rational number can not sum to a rational number.