MACM 101 — Discrete Mathematics I

Outline Solutions to Exercises on Set Theory and Relations

1. **Let** $A = \{x \in \mathbb{Z} \mid x = 5a + 2$ **for some integer** $a\}$,
   $B = \{y \in \mathbb{Z} \mid y = 10b - 3$ **for some integer** $b\}$,
   $C = \{z \in \mathbb{Z} \mid z = 10c + 7$ **for some integer** $c\}$,
   Prove or disprove each of the following statements:

   a. $A \subseteq B$ **b.** $B \subseteq A$ **c.** $B = C$

a. $A \subseteq B$. False. Indeed, $12 = 5 \cdot 2 + 2 \in A$ by setting $a = 2$. However, for any $x \in B$, $x + 3$ must be divisible by 10, while $12 + 3$ is not. Thus, $12 \notin B$.

b. $B \subseteq A$ True. Let $x \in B$, this means $x = 10b - 3$ for some $b \in \mathbb{Z}$. Then $x = 10b - 3 = 5 \cdot (2b) - 3 = 5 \cdot (2b - 1) + 2$. Thus, setting $a = 2b - 1 \in \mathbb{Z}$ we have $x = 5a + 2 \in A$.

c. $B = C$. True. Let $x \in B$, this means $x = 10b - 3$ for some $b \in \mathbb{Z}$. Then $x = 10(b - 1) + 7$, and setting $c = b - 1$ we obtain $x \in C$. Similarly, if $x \in C$ then $x = 10c + 7$ for some $c \in \mathbb{Z}$. Then $x = 10(c + 1) - 3$ and we have $x \in B$.

2. **Let the universe be the set** $\mathbb{R}$ **of all real numbers and let**
   $A = \{x \in \mathbb{R} \mid 0 < x \leq 2\}$, $B = \{x \in \mathbb{R} \mid 1 \leq x < 4\}$,
   $C = \{x \in \mathbb{R} \mid 3 \leq x < 9\}$.
   Find each of the following:

   a. $A \cup B$ **b.** $A \cap B$ **c.** $\overline{A}$ **d.** $A \cup C$
   e. $A \cap C$ **f.** $\overline{B}$ **g.** $\overline{A} \cap \overline{B}$
   h. $\overline{A} \cup \overline{B}$ **i.** $\overline{A \cap B}$ **j.** $\overline{A \cup B}$

   a. $A \cup B = \{x \in \mathbb{R} \mid 0 < x < 4\}$.
   b. $A \cap B = \{x \in \mathbb{R} \mid 1 \leq x \leq 2\}$
   c. $\overline{A}\{x \in \mathbb{R} \mid x \leq 0$ or $x > 2\}$
   d. $A \cup C\{x \in \mathbb{R} \mid 0 < x \leq 2$ or $3 \leq x < 9\}$
   e. $A \cap C = \emptyset$
   f. $\overline{B} = \{x \in \mathbb{R} \mid x < 1$ or $x \geq 4\}$
   g. $\overline{A} \cap \overline{B} = \{x \in \mathbb{R} \mid x \leq 0$ or $x \geq 4\}$
   h. $\overline{A} \cup \overline{B} = \{x \in \mathbb{R} \mid x < 1$ or $x > 2\}$
   i. $\overline{A \cap B} = \overline{A} \cup \overline{B} = \{x \in \mathbb{R} \mid x < 1$ or $x > 2\}$
   j. $\overline{A \cup B} = \overline{A} \cap \overline{B} = \{x \in \mathbb{R} \mid x \leq 0$ or $x \geq 4\}$

3. **Find** $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$**, where** $\mathcal{P}(A)$ **denotes the power set of** A**.**

   Note that $\mathcal{P}(\emptyset) = \{\emptyset\}$. Therefore $\mathcal{P}(\mathcal{P}(\emptyset)) = \mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$, and $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \mathcal{P}(\{\emptyset, \{\emptyset\}\}$, that is, $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

4. **Using laws of set theory show that**

$$\overline{(A \cup B) \cap C} = (\overline{A} \cup \overline{C}) \cap (\overline{B} \cup \overline{C}).$$

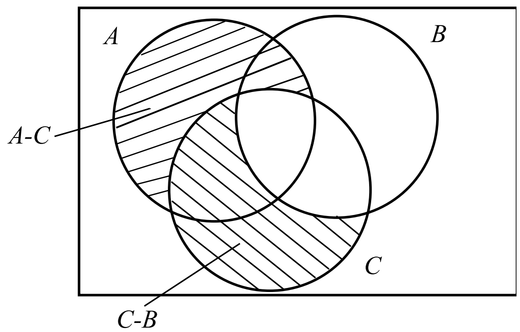
$$
\begin{aligned}
&\overline{(A \cup B) \cap C} \\
&\quad = \overline{(A \cap C) \cup (B \cap C)} && \text{distributive law} \\
&\quad = (\overline{A \cap C}) \cap (\overline{B \cap C}) && \text{DeMorgan's law} \\
&\quad = (\overline{A} \cup \overline{C}) \cap (\overline{B} \cup \overline{C}) && \text{DeMorgan's law}
\end{aligned}
$$

5. **Let A, B, and C be sets. Show that**

$$(A - C) \cap (C - B) = \emptyset.$$

Draw Venn diagrams for the left hand side expression.

Suppose $(A - C) \cap (C - B) \neq \emptyset$. Take an element $a \in (A - C) \cap (C - B)$. It belongs to both $A - C$ and $C - B$. From the first inclusion $a \in A - C$ it follows that $a \notin C$. From the second inclusion $a \in C - B$ it follows that $a \in C$. A contradiction.



6. **What can you say about sets A and B if we know that $A - B = A$? Explain.**

Sets A and B are disjoint, i.e. $A \cap B = \emptyset$. Indeed, suppose there is $a \in A \cap B$. Then $a \notin A - B$ for $a \in B$. On the other hand, $a \in A$; therefore, $A - B \neq A$, a contradiction.

7. **Prove that for all sets A, B, C, if $B \cap C \subseteq A$, then $(C - A) \cap (B - A) = \emptyset$.**

Suppose $a \in (C - A) \cap (B - A)$. Then $a \in C - A$ and $a \in B - A$. By definition of set difference, $a \in C$ and $a \in B$, that is, $a \in B \cap C \subseteq A$. Therefore $a \in A$, and so $a \notin C - A$ and $a \notin B - A$, a contradiction. Thus, $(C - A) \cap (B - A) = \emptyset$.

8. **Show that for any sets A and B**

$$A \triangle B = \overline{A} \triangle \overline{B}.$$

Draw Venn diagrams for both expressions.

Method 1. Use equality $A \triangle B = (A \cap \overline{B}) \cup (\overline{A} \cap B)$. Then

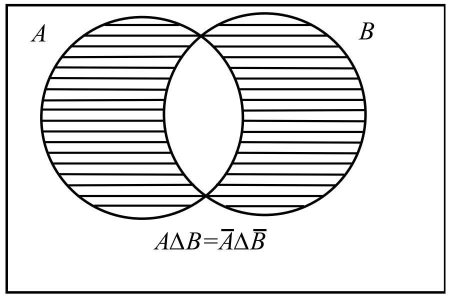
$$
\begin{aligned}
&\overline{A} \triangle \overline{B} \\
&\quad = (\overline{A} \cap \overline{\overline{B}}) \cup (\overline{\overline{A}} \cap \overline{B}) && \text{expression for symmetric difference} \\
&\quad = (\overline{A} \cap B) \cup (A \cap \overline{B}) && \text{complement law} \\
&\quad = A \triangle B && \text{expression for symmetric difference}
\end{aligned}
$$

Method 2. We prove that $A \triangle B \subseteq \overline{A} \triangle \overline{B}$ and $\overline{A} \triangle \overline{B} \subseteq A \triangle B$. In other words we show that if $x \in A \triangle B$ then $x \in \overline{A} \triangle \overline{B}$, and that if $x \in \overline{A} \triangle \overline{B}$ then $x \in A \triangle B$.

Suppose that $x \in A \triangle B$. There are two possibilities.
Possibility 1. $x \in A$ and $x \notin B$. Then $x \in \overline{B}$ and $x \notin \overline{A}$. Thus, $x \in \overline{B} - \overline{A}$.
Possibility 2. $x \in B$ and $x \notin A$. Then $x \in \overline{A}$ and $x \notin \overline{B}$. Thus, $x \in \overline{A} - \overline{B}$.

In both cases $x \in \overline{A} \triangle \overline{B}$.

The inclusion $\overline{A} \triangle \overline{B} \subseteq A \triangle B$ is proved in a similar way.



9. **Prove that**

$$A \times B \times (A \cap C) = (A \times B \times A) \cap (A \times B \times C).$$

Method 1. We have

$$\begin{aligned}
&A \times B \times (A \cap C) \\
&\quad = \{(a, b, c) \mid (a \in A) \land (b \in B) \land (c \in A \cap C)\} \\
&\quad = \{(a, b, c) \mid (a \in A) \land (b \in B) \land (c \in A \land c \in C)\} \\
&\quad = \{(a, b, c) \mid ((a \in A) \land (b \in B) \land (c \in A)) \land ((a \in A) \land (b \in B) \land (c \in C))\} \\
&\quad = \{(a, b, c) \mid (a \in A) \land (b \in B) \land (c \in A)\} \cap \{(a, b, c) \mid (a \in A) \land (b \in B) \land (c \in C)\} \\
&\quad = (A \times B \times A) \cap (A \times B \times C).
\end{aligned}$$

Method 2. We show that $A \times B \times (A \cap C) \subseteq (A \times B \times A) \cap (A \times B \times C)$, and that $(A \times B \times A) \cap (A \times B \times C) \subseteq A \times B \times (A \cap C)$.

$A \times B \times (A \cap C) \subseteq (A \times B \times A) \cap (A \times B \times C)$. Take an element (a, b, c) from $A \times B \times (A \cap C)$. Then $c \in A \cap C$, and hence $c \in A$ and $c \in C$. Since $a \in A, b \in B$, we have $(a, b, c) \in A \times B \times A$, and we have $(a, b, c) \in A \times B \times C$. Thus $(a, b, c) \in (A \times B \times A) \cap (A \times B \times C)$.

$(A \times B \times A) \cap (A \times B \times C) \subseteq A \times B \times (A \cap C)$. Take an element (a, b, c) from $(A \times B \times A) \cap (A \times B \times C)$. Then $(a, b, c) \in A \times B \times A$ and $(a, b, c) \in A \times B \times C$. This implies $a \in A, b \in B$; and also it implies that $c \in A \cap C$. Thus, $(a, b, c) \in A \times B \times (A \cap C)$.

10. **What is wrong with the following proof? You do not need to write down a formal inference or show a wrongly used rule of inference. Just point a step in the argument that is not valid and explain why.**

Theorem. For all sets A, B, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Proof. Suppose A and B are sets, and $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ or $x \in \overline{B}$ by definition of union. It follows that $x \notin A$ or $x \notin B$ by definition of complement, and so $x \notin A \cup B$ by definition of union. Thus $x \in \overline{A \cup B}$, and hence $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

That $x \notin A$ or $x \notin B$ does not imply that $x \notin A \cup B$. It is may be the case that $x \in A - B$ or $x \in B - A$.