MACM 101 — Discrete Mathematics I

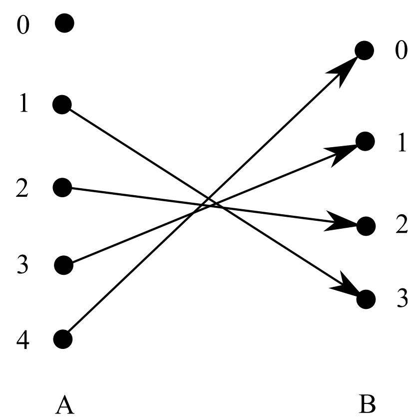
# Outline Solutions to Exercises on Set Theory and Relations

1. **Make a list of pairs, construct the matrix, and draw the graph of the relation $R$ from the set $A = \{0, 1, 2, 3, 4\}$ to the set $B = \{0, 1, 2, 3\}$ such that $(a, b) \in R$ if and only if $a + b = 4$.**

   The set of pairs $R = \{(1, 3), (2, 2), (3, 1), (4, 0)\}$; the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

   and the graph




2. **Let $R$ be a binary relation on the set of all positive integers given by $(x, y) \in R$ if and only if $xy \geq 1$. Determine if this relation reflexive, symmetric, transitive, or antisymmetric. Prove your answer.**

   Reflexivity. We need to check if $(x, x) \in R$, i.e. $x^2 \geq 1$ for all $x \in \mathbb{Z}^+$. This is clearly true.

   Symmetricity. The relation is symmetric, because if $(x, y) \in R$ then $xy \geq 1$, and so $yx \geq 1$ implying $(y, x) \in R$.

   Transitivity. $R$ is transitive. The easiest way to see that is to observe that $(x, y) \in R$ for any $x, y \in \mathbb{Z}^+$

   Antisymmetricity. $R$ is not antisymmetric. For instance, $(1, 2), (2, 1) \in R$, but $1 \neq 2$.

3. **Let $X$ be the set of all 4-bit strings (e.g. 0011,0101,1000, etc.). Define a relation $R$ on $X$ as $(s, t) \in R$ if and only if some substring of $s$ of length 2 is equal to some substring of $t$ of length 2. For example, $(0111, 0101) \in R$ because both 0111 and 0101 contain 01; however, $(1110, 0001) \notin R$ because 1110 and 0001 do not share a common substring of length 2. Is this relation reflexive, symmetric, transitive, or antisymmetric? Prove your answer.**

   We will denote bit strings as, say, $b_1b_2b_3b_4$, where $b_1, b_2, b_3, b_4$ are bits.

   Reflexivity. The relation is obviously reflexive: for any $b_1b_2b_3b_4$ the pair $(b_1b_2b_3b_4, b_1b_2b_3b_4) \in R$ because the two strings have all their 2-bit substrings equal.

   Symmetricity. The relation is symmetric. Indeed, if $(b_1b_2b_3b_4, c_1c_2c_3c_4) \in R$, that is, $b_1b_2b_3b_4$ and $c_1c_2c_3c_4$ have a common 2-bit substring then the same substring is common to $c_1c_2c_3c_4$ and $b_1b_2b_3b_4$ implying $(c_1c_2c_3c_4, b_1b_2b_3b_4) \in R$.

Transitivity. The relation is not transitive. Consider $a = 0000, b = 0011, c = 1111$. Then the substring 00 witnesses that $(a, b) \in R$ and the string 11 witnesses that $(b, c) \in R$. However, 0000 and 1111 have no common substrings, and so $(0000, 1111) \notin R$.

Antisymmetricity. The relation is not antisymmetric. Indeed, as we saw $(0000, 0011), (0011, 0000) \in R$ but $0000 \neq 0011$.

4. **Construct a relation on the set $\{1, 2, 3, 4\}$ that is reflexive, antisymmetric, and not transitive.**

   The relation $R = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (0, 1), (1, 2)\}$ will do (check!).

5. **Recall that binary relations are merely sets of pairs of elements of some set. Therefore if $R$ and $Q$ are binary relations on a same set $A$, their union $R \cup Q$ as sets of pairs is also a binary relation on $A$. Is it true that if $R$ and $Q$ are antisymmetric then $R \cup Q$ is also antisymmetric? Prove that it is or give a counterexample.**

   It is not true. Consider relations $R = \{(0, 1)\}$ and $Q = \{1, 0)\}$ on the set $A = \{0, 1\}$. Then both of them are antisymmetric, while $R \cup Q = \{(0, 1), (1, 0)\}$ is not.

6. **Let $R$ be the relation on $\mathbb{Z} \times \mathbb{Z}$, that is elements of this relation are pairs of pairs of integers, such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that $R$ is an equivalence relation.**

   We should prove that $R$ is reflexive, symmetric, and transitive.

   Reflexivity. For any $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, we have $a + b = b + a$.

   Symmetricity. Let $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$ be such that $a + d = b + c$. Then $c + b = d + a$, and therefore $((c, d), (a, b)) \in R$.

   Transitivity. Let $((a, b), (c, d)), ((c, d), (e, f)) \in R$, that is, $a + d = b + c$ and $c + f = d + e$. Adding this two equations up we get $a + d + c + f = b + c + d + e$, hence, $a + f = b + e$ implying $((a, b), (e, f)) \in R$.

7. **Let $f$ be a function from $A$ to $B$. Define a relation $R$ on $A$ by $(x, y) \in R$ if and only if $f(x) = f(y)$. Prove that $R$ is an equivalence relation.**

   We should prove that $R$ is reflexive, symmetric, and transitive.

   Reflexivity. Since $f(x) = f(x)$ for any $x \in A$, we have $(x, x) \in A$ for every $x \in A$.

   Symmetricity. If $(x, y) \in R$ then $f(x) = f(y)$, which implies $f(y) = f(x)$, and so $(y, x) \in R$.

   Transitivity. If $(x, y), (y, z) \in R$, then $f(x) = f(y)$ and $f(y) = f(z)$. This implies $f(x) = f(z)$, i.e. $(x, z) \in R$.

8. **Relation $R$ is given by matrix**

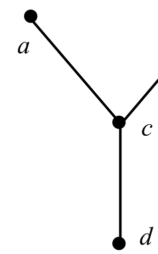
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

   **Is $R$ an order? If yes, what its minimal, maximal, least, and greatest elements are?**

   We need to check that this relation is reflexive, transitive, and anti-symmetric. Reflexivity can be easily seen from the matrix: every entry on the diagonal equals 1. Anti-symmetricity also can be seen from the matrix: for every off-diagonal entry that is equal to 1 the symmetric entry equals 0.

   To verify transitivity let us name the elements of the set on which the relation is given by $a, b, c, d$ (accordingly to the order of rows). Then the list of pairs of this relation is $R = \{(a, a), (b, b), (c, a), (c, b), (c, c), (d, a), (d, b), (d, c), (d, d)\}$. For every two pairs of the form $(x, y) \in R$ and $(y, z) \in R$ we must verify that $(x, z) \in R$. Note that $x$ can be equal to $y$, or $y$ can be equal to $z$, or even $x = y = z$. We proceed as follows. Take $(a, a)$ for $(x, y)$ then the second pair should start with $a$. There is only one such pair $(a, a)$. We have $(a, a), (a, a) \in R$, and have to check that the pair $(x, z) \in R$; that is $(a, a) \in R$. It is true. For the second pair, $(b, b)$ there is only one matching pair $(b, b)$, and they do not violate the transitivity condition. Then we check $(c, a)$ (and the matching pairs $(a, a)$), $(c, b)$, etc.

Once we proved that $R$ is an order we can draw its diagram (see next page). From the diagram we see that $R$




has one minimal element, $d$, which also the least element. $R$ also has two maximal elements $a$ and $b$, and no greatest element.

9. **Let $A = \{1, 2, 3, 4\}$, and let $R$ be a binary relation on $A \times A$ given by: $((a, b), (c, d)) \in R$ if and only if $a \leq c$ and $b \leq d$. Show that $R$ is an order and draw its diagram.**

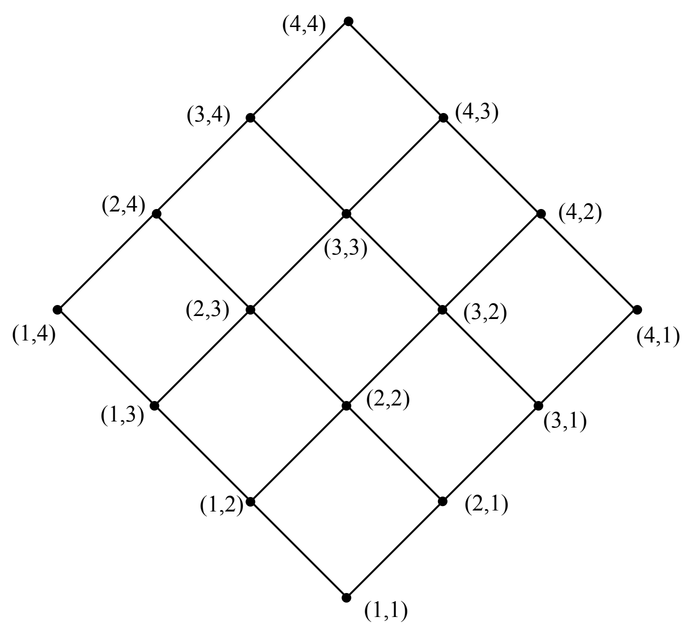
   Let us denote $B = A \times A$. This is a set of pairs: $B = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$. Show that $R$ is an order.

   *Reflexivity.* For any $(a, b) \in B$ we have $((a, b), (a, b)) \in R$ because $a \leq a$ and $b \leq b$.

   *Anti-Symmetricity.* Suppose that $((a, b), (c, d)) \in R$ and $((c, d), (a, b)) \in R$. Then from the first pair we obtain $a \leq c$ and $b \leq d$, while from the second one we get $c \leq a$ and $d \leq b$. Now, since $a \leq c$ and $c \leq a$, it follows that $a = c$. Similarly $b = d$. Thus $(a, b) = (c, d)$.

   *Transitivity.* Suppose that $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$. Then from the first pair we obtain $a \leq c$ and $b \leq d$, while from the second one we get $c \leq e$ and $d \leq f$. Now, since $a \leq c$ and $c \leq e$, it follows that $a \leq e$. Similarly $b \leq f$. Thus $((a, b), (e, f)) \in R$.

   Finally the diagram of this order looks as follows:



10. **Let $R$ be a relation that is symmetric and antisymmetric. Show that $R$ is transitive.**

If a relation $R$ is symmetric and antisymmetric then it is a subset of the equality relation, that is, if $(a, b) \in R$ then $a = b$. Indeed, if $(a, b) \in R$, then by symmetricity $(b, a) \in R$. Since $(a, b), (b, a) \in R$, antisymmetricity implies $a = b$. Now we prove transitivity. If $(a, b), (b, c) \in R$ then $a = b$ and $b = c$, thus, $a = c$. Therefore, $(a, c) = (a, b) \in R$.