MACM 101 — Discrete Mathematics I

Outline Solutions to Exercises on Functions and Induction

1. **Is the function $f\colon \mathbb{Z} \to \mathbb{Z}$ defined as $f(n) = n^2 + 2$ one-to-one?**

   No, because $f(2) = 2^2 + 2 = 6$ and $f(-2) = (-2)^2 + 2 = 6$.

2. **Determine whether or not the function $f\colon \mathbb{Z} \times (\mathbb{Z} - \{0\}) \to \mathbb{Z}$ is onto, if $f((m, n)) = \lfloor \frac{m}{n} \rfloor$.**

   It is onto. To show that we have to find, given an integer b, a pair (m, n) such that $\lfloor \frac{m}{n} \rfloor = b$. Clearly, we can choose $m = b$ and $n = 1$.

3. **Let f be a function from X to Y. For $A \subseteq X$, let also $f(A)$ denote the set $f(A) = \{f(a) \mid a \in A\}$. Prove that f is one-to-one if and only if**

   $$f(A \cap B) = f(A) \cap f(B)$$

   for any subsets $A, B \subseteq X$.

   Suppose first that f is not one-to-one. Then there are $a, b \in X$ such that $f(a) = f(b)$. Let $f(a) = f(b) = c \in Y$. Then for $A = \{a\}, B = \{b\}$ we have $f(A) = f(B) = \{c\}$, and so $f(A) \cap f(B) = \{c\}$. On the other hand, $f(A \cap B) = f(\emptyset) = \emptyset$.

   Then suppose that $f(A\cap B) \neq f(A)\cap f(B)$ for some $A, B \subseteq X$. If $a \in A\cap B$ then $f(a) \in f(A)\cap f(B)$. Thus, $f(A \cap B) \subseteq f(A) \cap f(B)$, always. Therefore, as $f(A \cap B) \neq f(A) \cap f(B)$, it must be that $f(A) \cap f(B) \not\subseteq f(A \cap B)$. Let $c \in f(A \cap B)$ but $c \notin f(A \cap B)$. Then for some $a \in A$ and $b \in B$ we have $c = f(a), c = f(b)$. If $a = b$ then $a \in A \cap B$ and $c \in f(A \cap B)$. Since this is not the case, $a \neq b$ and f is not one-to-one.

4. **Let f be a function from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z} \times \mathbb{Z}$ (that is, f maps pairs of integers to pairs of integers) given by $f(x, y) = (x + y, x - y)$. Is f one-to-one? onto? Prove or give a counterexample.**

   To check injectivity, suppose $f(x_1, y_1) = f(x_2, y_2)$. Then $x_1 + y_1 = x_2 + y_2$ and $x_1 - y_1 = x_2 - y_2$. Adding up the two equalities we get $2x_1 = 2x_2$, and this also implies $y_1 = y_2$. Thus, f is one-to-one.

   To check whether f is surjective, take $(a, b) \in \mathbb{Z}\times\mathbb{Z}$. We need to find $(x, y) \in \mathbb{Z}\times\mathbb{Z}$ such that $f(x, y) = (a, b)$, or, in other words,$x + y = a, x - y = b$. It is not hard to see that this is not always true, because $x + y$ and $x - y$ have to have the same parity, while a, b don't have to. For instance, if $a = 1 and b = 0$ then x must be equal to y and so $2x = 1$, which is impossible for an integer x.

5. **Given $g = \{(1, b), (2, c), (3, a)\}$, a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$, and $f = \{(a, \gamma), (b, \alpha), (c, \delta), (d, \gamma)\}$, a function from Y to $Z = \{\alpha, \beta, \gamma, \delta\}$, write $f \circ g$ as a set of ordered pairs.**

   $f \circ g = \{(1, \alpha), (2, \delta), (3, \gamma)\}$.

6. **Let $f(x) = ax + b$ and $g(x) = cx^2 + dx$, where a, b, c, and d are constants. Compute $f \circ g$ and $g \circ f$. Determine for which constants a, b, c, and d it is true that $f \circ g = g \circ f$.**

   First, compute $f \circ g$ and $g \circ f$:

   $$\begin{aligned} f \circ g(x) &= f(g(x)) = a \cdot g(x) + b = a(cx^2 + dx) + b = acx^2 + adx + b \\ g \circ f(x) &= g(f(x)) = c \cdot (f(x))^2 + d \cdot f(x) = c(ax + b)^2 + d(ax + b) = a^2cx^2 + (2abc + ad)x + b^2c + bd. \end{aligned}$$

   Now, let us express the condition $f \circ g = g \circ f$:

   $$\begin{aligned} (f \circ g)(x) &= (g \circ f)(x) \\ acx^2 + adx + b &= a^2cx^2 + (2abc + ad)x + b^2c + bd. \end{aligned}$$

   Two polynomials are equal for all values of x if and only their coefficients are equal. Therefore, the equality $f \circ g = g \circ f$ holds if and only if the constants a, b, and c satisfy the equality

   $$ac = a^2c, \qquad 2abc = 0, \qquad b = b^2c + bd.$$

Then, the second condition implies one of a, b, c must be 0. If $a = 0$ then from the third condition we obtain $b(bc + d - 1) = 0$. Therefore, either $b = 0$, c, d are any; or b, c are any and $d = -bc + 1$. If $b = 0$ then either c, d are any and $a = 0$ or $a = 1$; . Finally if $c = 0$ then a is any, and either $b = 0$ and d is any; or $b \neq 0$ is any, and $d = 1$.

7. **Let f, g be functions from positive integers to positive integers givem by formulas**

$$f(n) = n^2, \qquad g(n) = 2^n.$$

Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, $g \circ f$.

We have

$$\begin{aligned}
(f \circ f)(n) &= f(f(n)) = (f(n))^2 = (n^2)^2 = n^4 \\
(g \circ g)(n) &= g(g(n)) = 2^{g(n)} = 2^{2^n} \\
(f \circ g)(n) &= f(g(n)) = (g(n))^2 = (2^n)^2 = 2^{2n} \\
(g \circ f)(n) &= g(f(n)) = 2^{f(n)} = 2^{n^2}
\end{aligned}$$

8. **Given $f = \{(a, b), (b, a), (c, b)\}$, a function from $X = \{a, b, c\}$ to X.**

   (a) Write $f \circ f$ and $f \circ f \circ f$ as sets of ordered pairs.

   (b) Define

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \textbf{ times}}.$$

   Write f^{450} as a set of ordered pairs.

(a) $f \circ f = \{(a, a), (b, b), (c, a)\}$ and $f \circ f \circ f = \{(a, b), (b, a), (c, b)\}$

(b) It is not hard to see that for even n, $f^n = f \circ f$, and for odd n, $f^n = f \circ f \circ f$. Therefore $f^{450} = f \circ f = \{(a, a), (b, b), (c, a)\}$.

9. **If $f \circ g$ is one-to-one, does it follow that f is one-to-one?**

No, it does not. For example, let $A = \{a, b\}$, $B = \{a, b, c, d\}$, $C = \{a, b\}$, and let functions $g : A \to B$ and $f : B \to C$ be such that $g(a) = a, g(b) = b$, and $f(a) = f(c) = a$, $f(b) = f(d) = b$. Note that f is not one-to-one. However, $f \circ g : A \to C$ acts as follows: $(f \circ g)(a) = a$ and $(f \circ g)(b) = b$, that is, it is one-to-one.

10. **Show that the function $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{2\}$ defined by $f(x) = \frac{2x-3}{x-3}$ is a bijection, and find the inverse function.**

Note, first of all, that f is a function from $\mathbb{R} - \{3\}$, meaning that for any $a \in \mathbb{R} - \{3\}$ the number $f(a)$ is defined. Clearly, $f(3)$ is not defined. Next we prove that f is one-to-one by showing that if $f(a) = f(b)$ for some $a, b \in \mathbb{R} - \{3\}$ then $a = b$:

$$\begin{aligned}
f(a) &= f(b) \\
\frac{2a-3}{a-3} &= \frac{2b-3}{b-3} \\
(2a-3)(b-3) &= (a-3)(2b-3) \\
2ab - 6a - 3b + 9 &= 2ab - 6b - 3a + 9 \\
a &= b.
\end{aligned}$$

Thus, f is one-to-one.

To show that f is onto we take any $b \in \mathbb{R} - \{2\}$, and find $a \in \mathbb{R} - \{3\}$ such that $f(a) = b$.

$$\begin{aligned} f(a) &= b \\ \frac{2a-3}{a-3} &= b \\ 2a-3 &= ab - 3b \\ a &= \frac{3b-3}{b-2}. \end{aligned}$$

Thus, the required number a exists, and f is onto. Moreover, this computation gives us the inverse function:

$$f^{-1}(x) = \frac{3x-3}{x-2}.$$