MACM 101 — Discrete Mathematics I

Outline Solutions to Exercises on Induction

1. **Prove that the following equality is true for all integers** $n \geq 1$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

We use induction. Let $P(n)$ denote this equality for the integer n.
Basis case. For $n = 1$ we have $1^3 = \left[\frac{1\cdot(1+1)}{2}\right]^2$.
Inductive step. Suppose that $P(k)$ is true, that is,

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2.$$

Verify the equality for $k+1$.

$$\begin{aligned}
&1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 \\
&= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \\
&= \frac{1}{4}(k+1)^2(k^2+4k+4) \\
&= \frac{1}{4}(k+1)^2(k+2)^2 \\
&= \left[\frac{(k+1)(k+2)}{2}\right]^2,
\end{aligned}$$

as required.

2. **Verify that the following inequality is true for all integers** $n \geq 1$

$$\frac{1}{2n} \leq \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}.$$

We use induction. Let $P(n)$ denote this inequality for the integer n.
Basis case. For $n = 1$ we have $\frac{1}{2} \leq \frac{1}{2}$.
Inductive step. Suppose that $P(k)$ is true, that is,

$$\frac{1}{2k} \leq \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}.$$

Verify the inequality for $k+1$.

$$\begin{aligned}
\frac{1}{2(k+1)} &= \frac{1}{2k}\cdot\frac{2k}{2(k+1)} \\
&\leq \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}\cdot\frac{2k}{2(k+1)} \\
&= \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)\cdot 2k}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)\cdot 2(k+1)} \\
&= \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)\cdot(2k+1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)\cdot 2(k+1)},
\end{aligned}$$

as required.

3. **Use induction to prove that $11^n - 6$ is divisible by 5 for all integers $n \geq 1$.**

   We use induction. Let $P(n)$ denote this inequality for the integer n.
   Basis case. For $n = 1$ we have $11^1 - 6 = 5$ which is divisible by 5.

   Inductive step. Suppose that $P(k)$ is true, that is, $11^k - 6$ is divisible by 5, and so for some integer ℓ, $11^k - 6 = 5\ell$. Then $11^k = 5\ell + 6$. For $k + 1$ we have

   $$11^{k+1} - 6 = 11(5\ell + 6) - 6 = 5(11\ell) + 66 - 6 = 5(11\ell + 12),$$

   implying that $11^{k+1} - 6$ is divisible by 5.

4. **Let the numbers $c_1, c_2, \ldots$ be defined inductively as follows: $c_1 = 0$ and for every $n > 1$ the number c_n is given by $c_n = 4c_{\lfloor n/2 \rfloor} + n$.**

   Prove that $c_n \leq 4(n-1)^2$ for all integers $n \geq 1$.

   We use strong induction. Let $P(n)$ denote this inequality for the integer n.
   Basis case. For $n = 1$ we have $c_1 = 0 \leq 4(1-1)^2$.

   Inductive step. Suppose that $P(1), \ldots, P(k)$ are true, that is, $c_i \leq 4(i-1)^2$ for $1 \leq i \leq k$. For $k+1$ we have to prove that $c_{k+1} \leq 4((k+1)-1)^2 = 4k^2$. We consider two cases.

   If $k + 1$ is even then $c_{k+1} = 4c_{\frac{k+1}{2}} + k + 1$. Therefore

   $$\begin{aligned} c_{k+1} &= 4c_{\frac{k+1}{2}} + k + 1 \\ &\leq 4 \cdot 4 \left(\frac{k+1}{2} - 1 \right)^2 + k + 1 \\ &= 4 \cdot 4 \left(\frac{k-1}{2} \right)^2 + k + 1 \\ &= 4(k-1)^2 + k + 1 \\ &= 4k^2 - 7k + 5 \\ &\leq 4k^2, \end{aligned}$$

   provided $k > 0$.

   If $k + 1$ is odd, $c_{k+1} = 4c_{\frac{k}{2}} + k + 1$. Therefore

   $$\begin{aligned} c_{k+1} &= 4c_{\frac{k}{2}} + k + 1 \\ &\leq 4 \cdot 4 \left(\frac{k}{2} - 1 \right)^2 + k + 1 \\ &= 4k^2 - 16k + 16 + k + 1 \\ &= 4k^2 - 15k + 17 \\ &\leq 4k^2, \end{aligned}$$

   provided $k > 1$.

5. **Let the numbers $e_0, e_1, e_2, \ldots$ be defined inductively as follows:**

   $$\begin{aligned} &e_0 = 12, \qquad e_1 = 29 \\ &e_n = 5e_{n-1} - 6e_{n-2} \quad \textbf{for } n \geq 1. \end{aligned}$$

   Prove that $e_n = 5 \cdot 3^n + 7 \cdot 2^n$ for all integers $n \geq 1$.

We use strong induction. Let $P(n)$ denote this inequality for the integer n.
Basis case. For $n = 1$, we have $e_0 = 5 \cdot 3^0 + 7 \cdot 2^0 = 12$ and $e_1 = 5 \cdot 3^1 + 7 \cdot 2^1 = 15 + 14 = 29$.

Inductive step. Suppose that $P(1), \ldots, P(k)$ are true, that is, $e_i = 5 \cdot 3^i + 7 \cdot 2^i$ for $1 \le i \le k$. We need to prove $e_{k+1} = 5 \cdot 3^{k+1} + 7 \cdot 2^{k+1}$. we have

$$\begin{aligned} e_{k+1} &= 5e_k - 6e_{k-1} \\ &= 5(5 \cdot 3^k + 7 \cdot 2^k) - 6(5 \cdot 3^{k-1} + 7 \cdot 2^{k-1}) \\ &= 25 \cdot 3^k + 35 \cdot 2^k - 30 \cdot 3^{k-1} - 42 \cdot 2^{k-1} \\ &= 25 \cdot 3^k + 35 \cdot 2^k - 10 \cdot 3^k - 21 \cdot 2^k \\ &= 15 \cdot 3^k + 14 \cdot 2^k \\ &= 5 \cdot 3^{k+1} + 7 \cdot 2^{k+1}. \end{aligned}$$

6. **The kth harmonic number is defined to be**

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}.$$

Prove that harmonic numbers satisfy the equality

$$H_1 + H_2 + \cdots + H_n = (n+1)H_n - n$$

for all $n \in \mathbb{N}$.

We use induction. Let $P(n)$ denote this equality for the integer n.
Basis case. $P(1)$ means the equality $H_1 = 1 = (1+1) \cdot 1 - 1 = (1+1)H_1 - 1$, which is obviously true.
Inductive step. Suppose that $P(k)$ is true, that is,

$$H_1 + H_2 + \cdots + H_k = (k+1)H_k - k.$$

We have to prove $P(k+1)$:

$$H_1 + H_2 + \cdots + H_k + H_{k+1} = ((k+1)+1)H_{k+1} - (k+1).$$

Observe that $H_{k+1} = H_k + \frac{1}{k+1}$. We have

$$\begin{aligned} & H_1 + H_2 + \cdots + H_k + H_{k+1} \\ &= (k+1)H_k - k + H_{k+1} \\ &= (k+1)(H_k + \frac{1}{k+1}) - 1 - k + H_{k+1} \\ &= (k+1)H_{k+1} - (k+1). \end{aligned}$$

7. **Fibonacci numbers $F_1, F_2, F_3, \ldots$ are defined by the rule: $F_1 = F_2 = 1$ and $F_k = F_{k-2} + F_{k-1}$ for $k > 2$. Lucas numbers $L_1, L_2, L_3, \ldots$ are defined in a similar way by the rule: $L_1 = 1$, $L_2 = 3$ and $L_k = L_{k-2} + L_{k-1}$ for $k > 2$.**
Show that Fibonacci and Lucas numvers satisfy the following equality for all $n \ge 2$

$$L_n = F_{n-1} + F_{n+1}.$$

We use induction. Let $P(n)$ denote this equality for the integer n.
Basis case. $P(2)$ and $P(3)$ mean the equalities $L_2 = 3 = 1 + 2 = F_1 + F_3$ and $L_3 = 4 = 1 + 3 = F_2 + F_4$.
Inductive step. Suppose that $P(k-1)$ and $P(k)$ is true, that is,

$$L_{k-1} = F_{k-2} + F_k, \qquad L_k = F_{k-1} + F_{k+1}.$$

We have to prove $P(k+1)$:

$$L_{k+1} = F_k + F_{k+2}.$$

Using the definitions of L_n and F_n, and the induction hypothesis

$$\begin{aligned} L_{k+1} &= L_{k-1} + L_k \\ &= (F_{k-2} + F_k) + (F_{k-1} + F_{k+1}) \\ &= (F_{k-2} + F_{k-1}) + (F_k + F_{k+1}) \\ &= F_k + F_{k+2}. \end{aligned}$$

8. **The game of Chomp is played by two players. In this game, cookies are laid out on a rectangular grid. The cookie in the top left position is poisoned. The two players take turns making moves; at each move, a player is required to eat a remaining cookie, together with all cookies to the right and/or below (that is all the remaining cookies in the rectangle, in which the first cookie eaten is the top left corner). The loser is the player who has no choice but to eat the poisoned cookie. Prove that if the board is square (and bigger than 1×1) then the first player has a winning strategy.**

   Let the players are denoted by A and B, and A goes first. In his first move A eats the cookie that lies on the diagonal of the grid just next to the poisoned one. This way he leaves only one row and one column of cookies (with the poisoned cookie in the intersection). We will show using the principle of strong induction that in this configuration A wins. If the size of the grid is n, we denote this statement by $P(n)$.
   Basis step. Prove $P(2)$, that is for a 2×2 grid. Then B can eat either the non-poisoned cookie in the first row, or the non-poisoned cookie in the first column. In either case A can eat the remaining non-poisoned cookie leaving B no choice.
   Inductive step. Suppose that $P(i)$ is true for all $i \leq k$, that is, in the configuration when cookies are arranged in one row and one column of length i with the poisoned one in the intersection, A wins.Consider the position with a row and a column of length $k+1$. Then B can eat several cookies either from the row, or from the column. Suppose he eats $\ell > 0$ cookies from the row. So, after his move the position consists of a row of cookies of length $k+1-\ell$ and a column of of length $k+1$. Then A responds eating ℓ cookies from the column, and in the new position we have a row and a column of cookies of length $k+1-\ell$. By the inductive hypothesis $P(k+1-\ell)$ is true, and therefore A wins in this position.

9. **Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute rs. Show that no matter how you split the piles, the sum of the products computed at each step equals $\frac{n(n-1)}{2}$.**

   Let us denote by $P(n)$ the statement that the sum of products is $\frac{n(n-1)}{2}$.
   Basis step. Obviously, if we have a pile of 2 stones we have to split it once into piles with $r = s = 1$ stones. Thus the sum of products (there is only one product rs) equals 1, that proves $P(2)$.
   Inductive step. Suppose $P(2), P(3), \ldots, P(k)$ are true. Take a pile with $k+1$ stones and split it arbitrarily. We obtain two piles, let us denote the number of stones in the first pile by r and that in the second one by s. Clearly, $r, s \leq k$ and $r + s = k + 1$. Therefore, by the inductive hypothesis, the sum of products for the first pile equals $\frac{r(r-1)}{2}$, and for the second one — $\frac{s(s-1)}{2}$. Together with the first product rs this gives

$$rs + \frac{r(r-1)}{2} + \frac{s(s-1)}{2} = \frac{2rs + r^2 + s^2 - r - s}{2} = \frac{(r+s)^2 - (r+s)}{2} = \frac{(k+1)^2 - (k+1)}{2} = \frac{(k+1)k}{2}.$$

10. **A complete binary tree is a graph defined through the following recursive definition.**
    Basis step: **A single vertex is a complete binary tree.**
    Inductive step: **If T_1 and T_2 are disjoint complete binary trees with roots r_1, r_2, respectively, the the graph formed by starting with a root r, and adding an edge from r to each of the vertices r_1, r_2 is also a complete**

binary tree.
The set of leaves and the set of internal vertices of a complete binary tree can also be defined recursively.
Basis step: **The root r is a leaf of of the complete binary tree with exactly one vertex r. This tree has no internal vertices.**
Inductive step: **The set of leaves of the tree T built from trees T_1, T_2 is the union of sets of leaves of T_1 and the set of leaves of T_2. The internal vertices of T are the root r of T and the union of the set of internal vertices of T_1 and the set of internal vertices of T_2.**
Use structural induction to show that $\ell(T)$ the number of leaves of a complete binary tree T, is one more than $i(T)$, the number of internal vertices of T.

Let $P(T)$ denote the statement that the number $\ell(T)$ of leaves is less than $2^{h(T)}$.
Basis step. Let T contains only one vertex. Then by the basis step of the definition $\ell(T) = 1$ and $h(T) = 0$, so, $2^{h(T)} = 1$. Thus $P(T)$ is true.
Inductive step. Let T is built from trees T_1, T_2 as described in the definition and suppose that $P(T_1)$ and $P(T_2)$ are true. Again by the definition $\ell(T) = \ell(T_1) + \ell(T_2)$ and $h(T) = \max(h(T_1), h(T_2)) + 1$. Finally, we get

$$\ell(T) = \ell(T_1) + \ell(T_2) \leq 2^{h(T_1)} + 2^{h(T_2)} \leq 2 \cdot 2^{\max(h(T_1),h(T_2))} = 2^{\max(h(T_1),h(T_2))+1} = 2^{h(t)},$$

as required.