Solutions to Week 11 Tutorial — MACM 101 (Spring 2025)

Problem 1. Which positive integers less than 30 are relatively prime with 30?

Answer.

Since

$$30 = 2 \cdot 3 \cdot 5,$$

a positive integer is relatively prime with 30 exactly when it is not divisible by 2, 3, or 5. Therefore, the positive integers less than 30 that are relatively prime with 30 are

$$1,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29.$$

Problem 2. For positive integers a, b and $d = \gcd(a, b)$, prove that

$$\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1.$$

Answer.

Let

$$g = \gcd\left(\frac{a}{d}, \frac{b}{d}\right).$$

Then g divides both a/d and b/d, so dg divides both a and b. Since d is the greatest positive common divisor of a and b, we must have $dg \leq d$. Thus $g \leq 1$. Since g is positive, $g = 1$.

Problem 3. Let n be a positive integer. Prove that $\gcd(n, n+2)$ equals 1 or 2.

Answer.

Let $d = \gcd(n, n+2)$. Since d divides both n and $n+2$, it also divides their difference:

$$d \mid (n+2) - n = 2.$$

The only positive divisors of 2 are 1 and 2. Therefore,

$$\gcd(n, n+2) \in \{1, 2\}.$$

In fact, the gcd is 2 when n is even and 1 when n is odd.

Problem 4. Find the greatest common divisor of 168 and 456, and also find integers u, v such that

$$\gcd(168, 456) = 168u + 456v.$$

Answer.

Using the Euclidean algorithm,

$$\begin{aligned} 456 &= 2 \cdot 168 + 120, \\ 168 &= 1 \cdot 120 + 48, \\ 120 &= 2 \cdot 48 + 24, \\ 48 &= 2 \cdot 24. \end{aligned}$$

Therefore,

$$\gcd(168, 456) = 24.$$

Working backward,

$$\begin{aligned} 24 &= 120 - 2 \cdot 48 \\ &= 120 - 2(168 - 120) \\ &= 3 \cdot 120 - 2 \cdot 168 \\ &= 3(456 - 2 \cdot 168) - 2 \cdot 168 \\ &= -8 \cdot 168 + 3 \cdot 456. \end{aligned}$$

Thus, one possible choice is

$$u = -8, \qquad v = 3.$$

Problem 5. Find the greatest common divisor and the least common multiple of 630 and 40452.

Answer.

The Euclidean algorithm gives

$$\begin{aligned} 40452 &= 64 \cdot 630 + 132, \\ 630 &= 4 \cdot 132 + 102, \\ 132 &= 1 \cdot 102 + 30, \\ 102 &= 3 \cdot 30 + 12, \\ 30 &= 2 \cdot 12 + 6, \\ 12 &= 2 \cdot 6. \end{aligned}$$

Hence

$$\gcd(630, 40452) = 6.$$

Using $\gcd(a, b)\,\mathrm{lcm}(a, b) = ab$, we obtain

$$\begin{aligned} \mathrm{lcm}(630, 40452) &= \frac{630 \cdot 40452}{6} \\ &= 4\,247\,460. \end{aligned}$$

Problem 6. Find the prime factorization of
(a) 148500; (b) 7114800; (c) 7882875.

Answer.

(a) We have

$$\begin{aligned} 148500 &= 1485 \cdot 100 \\ &= (3^3 \cdot 5 \cdot 11)(2^2 \cdot 5^2) \\ &= 2^2 \cdot 3^3 \cdot 5^3 \cdot 11. \end{aligned}$$

(b) Since $7114800 = 71148 \cdot 100$ and

$$71148 = 4 \cdot 17787 = 2^2 \cdot 3 \cdot 77^2,$$

we get

$$7114800 = 2^4 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot 11^2.$$

(c) We have

$$\begin{aligned} 7882875 &= 63063 \cdot 125 \\ &= (3^2 \cdot 7^2 \cdot 11 \cdot 13)5^3 \\ &= 3^2 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13. \end{aligned}$$

Problem 7. How many positive divisors are there for

$$n = 2^{14}3^9 5^8 7^{10} 11^3 13^5 37^{10}?$$

Answer.

Every positive divisor of n has the form

$$2^a 3^b 5^c 7^d 11^e 13^f 37^g,$$

where

$$0 \le a \le 14, \quad 0 \le b \le 9, \quad 0 \le c \le 8, \quad 0 \le d \le 10,$$

and similarly $0 \le e \le 3$, $0 \le f \le 5$, and $0 \le g \le 10$. Thus the number of positive divisors is

$$(14+1)(9+1)(8+1)(10+1)(3+1)(5+1)(10+1) = 3\,920\,400.$$

Problem 8. A perfect square is a number n such that $n = k^2$ for some integer k. Determine the smallest perfect square that is divisible by $7!$.

Answer.

The prime factorization of $7!$ is

$$7! = 5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7.$$

In a perfect square, every exponent in the prime factorization must be even. The smallest way to make the exponents of 5 and 7 even is to multiply by $5 \cdot 7$. Therefore, the smallest required square is

$$\begin{aligned} 7! \cdot 5 \cdot 7 &= 2^4 \cdot 3^2 \cdot 5^2 \cdot 7^2 \\ &= 420^2 \\ &= 176400. \end{aligned}$$

Problem 9. Show that if $2^n - 1$ is prime, then n is prime.

Hint: Use the identity

$$a^x - 1 = (a-1)(a^{x-1} + a^{x-2} + \cdots + a + 1).$$

Answer.

We prove the contrapositive. Suppose that n is composite. Then $n = rs$ for some integers $r, s > 1$. Applying the given identity with $a = 2^r$ and $x = s$, we obtain

$$\begin{aligned} 2^n - 1 &= (2^r)^s - 1 \\ &= (2^r - 1)\bigl(2^{r(s-1)} + 2^{r(s-2)} + \cdots + 2^r + 1\bigr). \end{aligned}$$

Both factors are greater than 1, so $2^n - 1$ is composite. Therefore, if $2^n - 1$ is prime, then n must be prime.