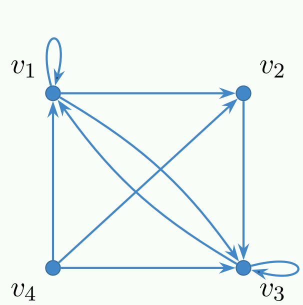
Problems to Week 13 Tutorial — MACM 101 (Summer 2026)

Problem 1. Draw a graph with the adjacency matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

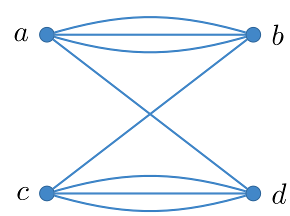
Answer.

We use the convention that the entry in row i and column j is the number of directed edges from v_i to v_j. One possible drawing is



For example, the first row gives the loop at v_1 and the edges $v_1 \to v_2$ and $v_1 \to v_3$. The zero fourth column means that no edge enters v_4.

Problem 2. Represent the following graph using adjacency lists, an adjacency matrix, and an incidence matrix.



Answer.

Repeated entries in an adjacency list record parallel edges. Thus

$$a : (b, b, b, d), \quad b : (a, a, a, c),$$
$$c : (d, d, d, b), \quad d : (c, c, c, a).$$

With the vertex order a, b, c, d, the adjacency matrix is

$$A = \begin{pmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{pmatrix}.$$

Let e_1, e_2, e_3 be the three ab-edges, let e_4, e_5, e_6 be the three cd-edges, and let $e_7 = ad$ and

$e_8 = bc$. The incidence matrix is

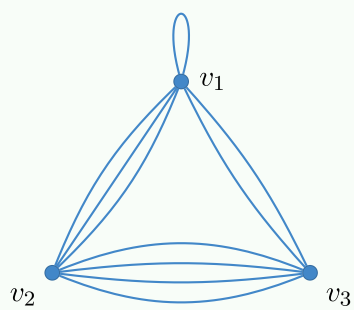
$$\begin{array}{c|cccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ \hline a & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ b & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ c & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ d & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}.$$

Problem 3. Draw an undirected multigraph represented by the adjacency matrix

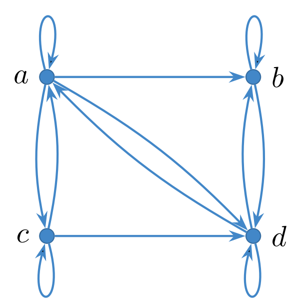
$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{pmatrix}.$$

Answer.

The diagonal entry 1 gives one loop at v_1. The other entries give three v_1v_2-edges, two v_1v_3-edges, and four v_2v_3-edges.



Problem 4. Represent the following directed multigraph using adjacency lists, an adjacency matrix, and an incidence matrix.



Answer.

We list outgoing neighbours. A loop therefore places the vertex itself in its list:

$$\begin{aligned} &a : (a, b, c, d), \quad &&b : (b, d), \\ &c : (a, c, d), &&d : (a, b, d). \end{aligned}$$

With vertex order a, b, c, d, the adjacency matrix is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

For the incidence matrix, use -1 at the tail and $+1$ at the head. Let e_1, e_2, e_3, e_4 be the loops at a, b, c, d, respectively, and set

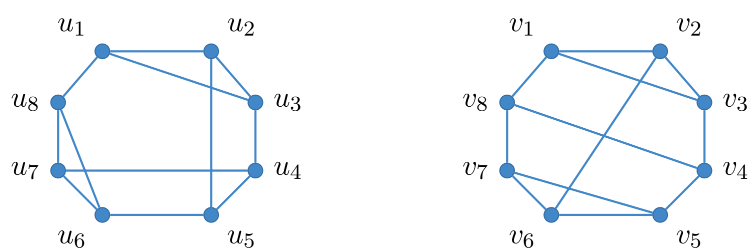
$$e_5 = a \to b, \quad e_6 = c \to d, \quad e_7 = a \to c, \quad e_8 = c \to a,$$
$$e_9 = b \to d, \quad e_{10} = d \to b, \quad e_{11} = a \to d, \quad e_{12} = d \to a.$$

A loop has the same head and tail, so in the incidence column we put a ± 1. Thus

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
a	± 1	0	0	0	-1	0	-1	1	0	0	-1	1
b	0	± 1	0	0	1	0	0	0	-1	1	0	0
c	0	0	± 1	0	0	-1	1	-1	0	0	0	0
d	0	0	0	± 1	0	1	0	0	1	-1	1	-1

.

Problem 5. Use graph parameters to determine whether the following graphs are isomorphic.



Answer.

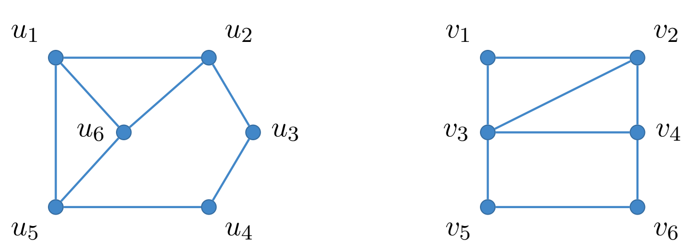
Both graphs have 8 vertices, 12 edges, and degree sequence $(3, 3, 3, 3, 3, 3, 3, 3)$. They are in fact isomorphic. One isomorphism is

u_i	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
$\varphi(u_i)$	v_2	v_1	v_3	v_4	v_8	v_7	v_5	v_6

.

Checking the outer-cycle edges and the four chords shows that every edge is sent to an edge, so φ is an isomorphism.

Problem 6. Use graph parameters to determine whether the following graphs are isomorphic.



Answer.

The degree sequences are different:

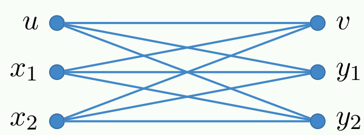
$$(3, 3, 3, 3, 2, 2) \qquad \text{and} \qquad (4, 3, 3, 2, 2, 2).$$

In particular, the second graph has a vertex of degree 4, while the first does not. Since an isomorphism preserves degrees, the graphs are not isomorphic.

Problem 7. Find the number of paths of lengths $2, 3, 4$, and 5 between any two adjacent vertices in $K_{3,3}$.

Answer.

The graph $K_{3,3}$ is the following (the label of vertices doesn't matter too much).



Let the two parts be

$$L = \{u, x_1, x_2\}, \qquad R = \{v, y_1, y_2\}.$$

Even-length paths cannot go from $u \in L$ to $v \in R$, because an even path starts and ends at the same side of the graph.
A path of length 3 has the form

$$u \longrightarrow y_i \longrightarrow x_j \longrightarrow v,$$

giving $2 \cdot 2 = 4$ choices.
A path of length 5 has the form

$$u \longrightarrow y_i \longrightarrow x_j \longrightarrow y_k \longrightarrow x_\ell \longrightarrow v.$$

There are two choices each for y_i and x_j, after which y_k and x_ℓ are forced. Thus, there are also $2 \cdot 2 = 4$ choices for paths of length 5 as well.

Problem 8. Show that in every simple graph there is a path from every vertex of odd degree to some other vertex of odd degree.

Answer.

Let v have odd degree, and consider the connected component C containing v. By the handshaking lemma, C has an even number of odd-degree vertices. Since v is one of them, C contains at least one other odd-degree vertex w. The vertices v and w lie in the same connected component, so there is a path from v to w.

Problem 9. How many nonisomorphic simple graphs with 4 vertices are there?

Answer.

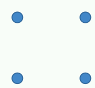
Classify the graphs by their number m of edges. Taking complements shows that the counts for m and $6-m$ are equal. The counts are

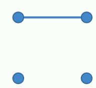
m	0	1	2	3	4	5	6
number of types	1	1	2	3	2	1	1

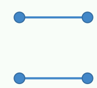
and hence the total is

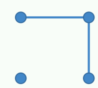
$$1+1+2+3+2+1+1=11.$$

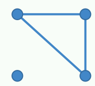
The following are representatives of the eleven types.

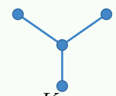

0 edges

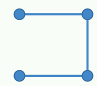

$K_2 \cup 2K_1$

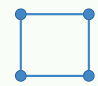

$2K_2$

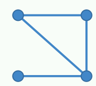

$P_3 \cup K_1$

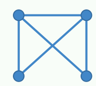

$K_3 \cup K_1$

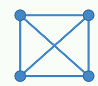

$K_{1,3}$


P_4

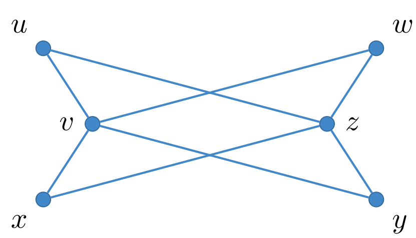

C_4


paw graph


$K_4 - e$


K_4

Problem 10. Is the following graph bipartite?



Answer.

Yes. A bipartition is

$$\{v,z\} \qquad \text{and} \qquad \{u,x,w,y\}.$$

Each of u, x, w, y is adjacent to both v and z, and there are no other edges. Thus the graph is in fact $K_{2,4}$.